Diffusion in Plasmas

So far, whenever we discussed about plasmas, we have assumed they are perfectly or mostly controlled.

In reality, however, whenever plasmas' boundaries have density gradient, they tend to diffuse in such a way from higher density region toward lower density region.

Magnetic field is commonly used, such as in Tokamak, to counter this diffusion and keep plasmas in the desired region.

Microscopically, diffusion occurs because of collisions inside plasmas.

We have already discussed two types of charge particle collisions: small and large angle collisions. Both are named Coulomb collision, but in plasmas, we conclude that small angle collisions dominate.

Nevertheless, we have not consider collisions with neutral atoms.

In general though, collisions are characterized by collision frequency $\nu = n\sigma v = v/\lambda$, or collision time $\tau = 1/\nu$

Governing Equation of Diffusion Process

Recall continuity equation

$$\frac{\partial n}{\partial t} + \underline{\nabla} \bullet n\underline{v} = 0$$

It is useful when discussing about diffusion to replace the term $n\underline{v}$ with flux $\underline{\Gamma}$ so that

$$\frac{\partial n}{\partial t} + \underline{\nabla} \bullet \underline{\Gamma} = 0$$

We can set

$$\boxed{\underline{\Gamma} = -D\underline{\nabla}n}$$

where D = diffusion coefficient [distance ^{2} / time or L^2/T] and is always positive. The reason behind this should be easy to understand. Velocity $\underline{v}$ in diffusion occurs as a result of density gradient.

Consequently, diffusion equation can be written as

$$\boxed{\frac{\partial}{\partial t} n(x,t) = D\nabla^2 n(x,t)}$$

where $T \sim L^2/D$ is the time scale for n to diffuse over a distance L.

T is related to the confinement time.

Let's now prove for D from scaling point of view.

Let

$w(x,t)$ = probability of a particle that starts at $x = 0$ and $t = 0$ is located at x after time t.

Δx = step size

Δt = time a particle takes to move distance Δx

We can write

$$w(x,t+\Delta t)=\frac{1}{2}w(x+\Delta x,t)+\frac{1}{2}w(x-\Delta x,t)$$

Then

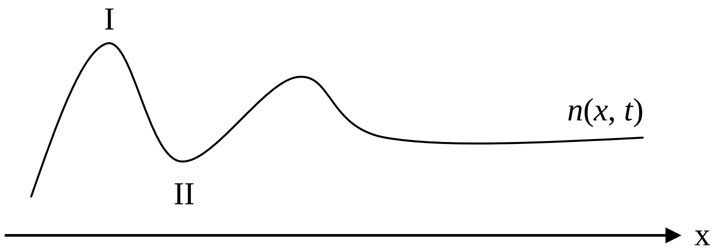
$$\begin{aligned} w(x,t+\Delta t)-w(x,t) &= \frac{1}{2}\{w(x+\Delta x,t)-w(x,t)\}-\frac{1}{2}\{w(x,t)-w(x-\Delta x,t)\} \\ &= \frac{1}{2}\left\{(\Delta x)\frac{\partial w}{\partial x}+\frac{1}{2}(\Delta x)^2\frac{\partial^2 w}{\partial x^2}+\ldots\right\} \\ &\quad -\frac{1}{2}\left\{(\Delta x)\frac{\partial w}{\partial x}-\frac{1}{2}(\Delta x)^2\frac{\partial^2 w}{\partial x^2}+\ldots\right\} \\ &= \frac{(\Delta x)^2}{2}\frac{\partial^2 w}{\partial x^2} \end{aligned}$$

Consequently,

$$\frac{\partial w}{\partial t}=\left[\frac{1}{2}\frac{(\Delta x)^2}{\Delta t}\right]\frac{\partial^2 w}{\partial x^2}\equiv D\frac{\partial^2 w}{\partial x^2}$$

Thus $D \sim (\Delta x)^2/(\Delta t) = \dfrac{(\text{step size})^2}{(\text{time between random events})}$.

Suppose $n(x, t)$ looks like the following



From $\frac{\partial}{\partial t}n(x,t) = D\frac{\partial^2}{\partial x^2}n(x,t)$, we can see that

Zone I: $\frac{\partial^2 n}{\partial x^2} < 0$, but $D > 0$. Then $\frac{\partial n}{\partial t} < 0$, which means that *n* decreases at later time.

Zone II: $\frac{\partial^2 n}{\partial x^2} > 0$, and $D > 0$. Then $\frac{\partial n}{\partial t} > 0$, which means that *n* increases at later time.

In both cases, *n* at point *x* changes in such a way that they try to get back to the equilibrium value by diffusing away or toward point *x*. In other words, *n* tends to spread out.

For example, let $n(x,0) = \delta(x)$ then $n(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-x^2/(4Dt)}$

The density starts out as a delta function, then as *t* increases, density becomes Gaussian function.

In time *t*, the diffusion distance scale as $\sqrt{4Dt} \sim L$

To diffuse over distance L, require time on order of L^2/D

There are 2 regimes of diffusions which we shall consider

- Diffusion in fully ionized plasmas. In this case, collisions are among charge particles.
- Diffusion in weakly ionized plasmas. In this case, collisions among charge particles will be rare. Majority of collisions will be between charge particles and neutral atoms.

Diffusion in Fully Ionized Plasmas

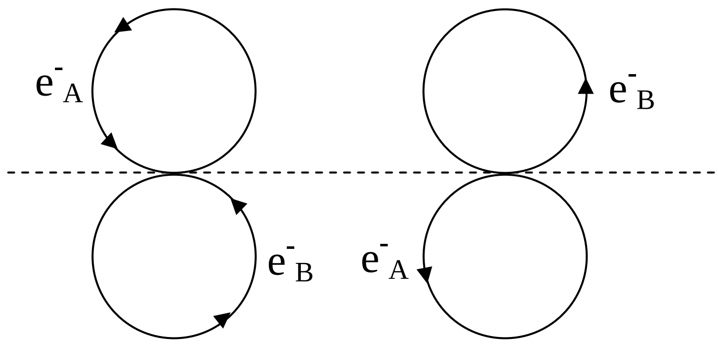
Coulomb collision in the main process that causes fully ionized plasma to diffuse.

We have calculated before the collision frequencies of charged particle collisions. That is, we found that

$$\nu \propto \frac{n}{T^{3/2}}$$

There are, however, distinctions between collisions among opposite charged particles and collision among same charged particles.

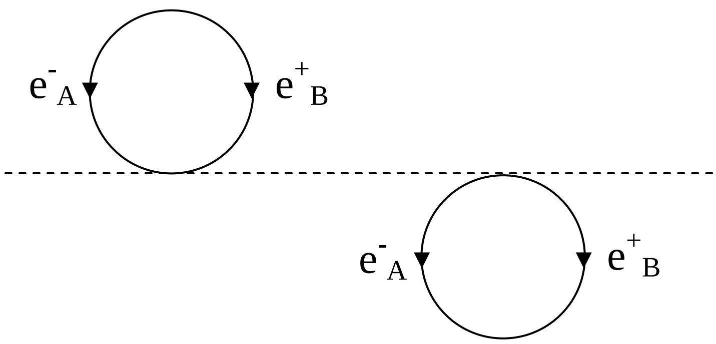
Consider first,



Before collision　　　After collision

When electron A collides with electron B with, they merely switch places with no diffusion path. Thus, electron – electron collision does not contribute appreciable diffusion.

Consider now another case,



Before collision　　　After collision

When electron A collides with positron B, both of their guiding centers change in the same direction. Thus electron – positron, or electron – ion, collision contributes to diffusion.

This is why we only consider electron – ion collision when we calculate diffusion coefficient.

How is *D*, a macroscopic quantity, related to microscopic process (e.g. collisions) in fully ionized plasma?

⇨ Claim

$$D = \frac{(\text{step size})^2}{(\text{time between random events})}$$
$$= \frac{r_{Le}^2}{\tau_{ei}}$$
$$= \nu_{ei} r_{Le}^2$$
$$= \nu_{ei} \left(\frac{m v_\perp}{eB} \right)^2$$

Since $m v_\perp^2 = KT$, then

$$\boxed{D = \nu_{ei} \frac{m}{e^2} \frac{KT}{B^2} = \frac{\eta n KT}{B^2}}$$

where we define **specific resistivity**

$$\eta = \frac{\nu_{ei} m}{n e^2}$$

We can see that $\eta \propto T^{-3/2}$, thus $D \propto \dfrac{nT^{-1/2}}{B^2}$. This is known as **classical diffusion coefficient**.