Last Time

- Diffusion in plasma: the main reason why we need to control it (i.e. using magnetic field)
- Diffusion equation, flux, diffusion coefficient scaling
  - $\frac{\partial}{\partial t} n(x,t) = D \nabla^2 n(x,t)$
  - $\underline{\Gamma} = -D \underline{\nabla} n$
  - $D \sim L^2 / T$
- Diffusion in fully ionized plasma vs. weakly ionized plasma
- Diffusion in fully ionized plasma
  - $\nu \propto \frac{n}{T^{3/2}}$ => Coulomb collision frequency
  - $D \propto \frac{n T^{-1/2}}{B^2}$ => Classical diffusion coefficient

Bohm Diffusion

We have derived classical diffusion coefficient last time.

Unfortunately, the classical diffusion coefficient does not work well in experiment because of the imperfection of the confinement.

A semiempirical diffusion coefficient is formularized, and found to work much better in experiment. It is known has **Bohm diffusion coefficient**:

$$D_B = \frac{1}{16}\frac{KT_e}{eB}$$

Here we can see that

- D_B is proportional to T instead of $T^{-1/2}$
- D_B is proportional to $1/B$ instead of $1/B^2$
- D_B does not depend on n!
- Plasma that diffuses with this process decays exponentially with time rather than linearly with time. (I will show this later)

As a result, Bohm diffusion coefficient is always larger than classical diffusion coefficient, sometimes by 4 order of magnitude!

The T and B dependences in D_B are due to ExB drift.

We know that $\Gamma = nv$. Substituting v by v_{ExB}, then

$$\Gamma \propto n\frac{E}{B}$$

The value of electric field can be roughly estimated by assuming maximum potential in the plasma. Because of Debye shielding,

$$e\phi_{\max} \approx KT_e$$

and so if characteristic length of the plasma is R, then

$$E_{\max} \approx \frac{\phi_{\max}}{R} \approx \frac{KT_e}{eR}$$

Consequently,

$$\Gamma \propto \frac{n}{R}\frac{KT_e}{eB} = -\gamma\frac{KT_e}{eB}\nabla n = -D_B\nabla n$$

The time constant in a cylindrical column of radius R and length L for Bohm diffusion is given as

$$\tau \approx \frac{N}{dN/dt} = \frac{n\pi R^2 L}{\Gamma 2\pi RL} = \frac{nR}{2\Gamma} = \frac{nR}{2D_B dn/dr} \approx \frac{nR}{2D_B n/R} = \frac{R^2}{2D_B}$$

This is also called Bohm time, and can be viewed as confinement time.

There are many experiments which have confirmed Bohm diffusion

[Figure 5-20 Chen]

Diffusion in weakly ionized plasma

Since in this case, charged particle collisions will be rare due to the small number of charged particles. Majority of collisions will be between charged particle and neutral particle.

Since neutron has no charge, collision between neutron and charged particle is just like collision between uncharged particles. That is, classical mechanic applies.

Imagine a slab with neutral atoms spreading inside

[Figure 5-2 Chen]

- Charged particles are coming in from the left
- Assume that the particles are electrons
- Flux of electrons = Γ = number of electrons per unit area per unit time
- There are n_n neutral atoms
- Neutral atom's cross-sectional size = σ = collision cross section
- Slab thickness = dx
- Slab area = A

Then

- Number of neutral atoms inside the slab = $n_n A\ dx$
- Total area blocked by neutral atoms in the slab = $n_n A\ dx\ \sigma$
- Total area unblocked by neutral atom = $A - n_n A\ dx\ \sigma = A(1 - n_n\ dx\ \sigma)$

- Fraction of unblocked area = $1 - n_n\, dx\, \sigma$
- Assume that when an electron hits a neutral atom, it either stops or is bounced back (most likely it gets bounced back because of the difference in mass).
- Flux of electrons which pass through the slab $\Gamma(1 - n_n\, dx\, \sigma) = \Gamma'$
- Thus

$$\Gamma' - \Gamma = -n_n \sigma \Gamma dx$$

$$\frac{d\Gamma}{dx} = -n_n \sigma \Gamma$$

Consequently, for slab of thickness x,

$$\Gamma = \Gamma_0 e^{-n_n \sigma x} = \Gamma_0 e^{-x/\lambda_m},$$

where $\lambda_m = 1/n_n \sigma$ = mean free path.

- As with before, we can then write

Collision time $\tau = \lambda_m / v$ = time between collision

Collision frequency $\nu = 1/\tau = n_n \sigma v$

Now recall fluid equation with collision

$$mn\frac{d\underline{v}}{dt} = mn\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v} \bullet \underline{\nabla})\underline{v}\right] = \pm en\underline{E} - \underline{\nabla} p - mn\nu\underline{v}$$

If we look at steady state, and assume that v is small (or ν is large) such that the fluid sees constant E and ∇p during a collision period, then the left-hand-side of the equation is zero. Consequently,

$$0 = \pm en\underline{E} - \underline{\nabla} p - mn\nu\underline{v}$$

$$\underline{v} = \pm\frac{e}{m\nu}\underline{E} - \frac{KT}{m\nu}\frac{\underline{\nabla} n}{n}$$

We can define

$$\mu \equiv \frac{|q|}{m\nu} \text{ = Mobility}$$

$$D \equiv \frac{KT}{m\nu} \text{ = Diffusion coefficient}$$

and thus,

$$\mu = \frac{|q|D}{KT}$$

Since we have stated before that

$$\underline{\Gamma} = n\underline{v}$$

then

$$\underline{\Gamma} = \pm\mu n\underline{E} - D\underline{\nabla}n$$

and in the absence of electric field E, we recover Fick's law, $\underline{\Gamma} = -D\underline{\nabla}n$ which we saw in the last lecture.

Certainly, if the incoming particles are uncharged, then we have zero mobility, and also recover Fick's law from the equation above.

Notice that the diffusion coefficient here depends on $T^{1/2}$ instead of $T^{-1/2}$ as in the case of classical diffusion coefficient for fully ionized gas. The reason is because of the velocity dependence of the Coulomb cross section.

Ambipolar diffusion

In a weakly ionized plasma, both electron and ion are acting species. They have different μ and different *D*.

Their Γ will be initially different, but eventually become the same because of the electric field setup between them.

That is, electron which has higher mobility and diffusion rate due to lower mass will slow down while ion which has lower mobility and diffusion rate will speed up. Then, we can set

$$\mu_i n\underline{E} - D_i\underline{\nabla}n = -\mu_e n\underline{E} - D_e\underline{\nabla}n$$

so that

$$\underline{E} = \left(\frac{D_i - D_e}{\mu_i + \mu_e}\right)\frac{\underline{\nabla}n}{n}$$

Thus, if we replace *E* in electron's or ion's flux equations, we then get

$$\begin{aligned}\Gamma &= -\mu_e n\left(\frac{D_i - D_e}{\mu_i + \mu_e}\right)\frac{\underline{\nabla}n}{n} - D_e\underline{\nabla}n \\ &= \left(\frac{-\mu_e D_i + \mu_e D_e - \mu_i D_e - \mu_e D_e}{\mu_i + \mu_e}\right)\underline{\nabla}n \\ &= -\left(\frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e}\right)\underline{\nabla}n\end{aligned}$$

Thus, we can define equivalent diffusion coefficient

$$D_a \equiv \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e}$$

which is sometimes known as **ambipolar diffusion coefficient**. This diffusion coefficient is larger than the ion diffusion coefficient, but lower than the electron diffusion coefficient.

Since the diffusion coefficient is the total coefficient which includes effect of electric field, we can then treat this diffusion as if there is no electric field. Thus, Fick's law can be applied, and consequently,

$$\frac{\partial n}{\partial t} = D_a \nabla^2 n$$

Diffusion in a slab geometry

For a slab geometry, we can write

$$n(\underline{r},t) = n(x,t) = S(x)T(t)$$

where $S(x)$ is only a function of x and $T(t)$ is only a function of t. Substitute this into the diffusion equation, we then have

$$S\frac{dT}{dt} = D_a T\nabla^2 S$$
$$\frac{1}{T}\frac{dT}{dt} = \frac{D_a}{S}\frac{d^2S}{dx^2}$$

Both sides depend on different variables. When t changes, the right-hand-side should not change. When x changes, the left-hand-side should not change. Thus, they both must equal to a constant, say $-1/\tau$. Then,

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{\tau}$$
$$\frac{dT}{dt} = -\frac{T}{\tau}$$
$$T = T_0 e^{-t/\tau}$$

and

$$\frac{D_a}{S}\frac{d^2S}{dx^2} = -\frac{1}{\tau}$$
$$\frac{d^2S}{dx^2} = -\frac{S}{D_a\tau}$$
$$S = A\cos\left(\frac{x}{\sqrt{D_a\tau}}\right) + B\sin\left(\frac{x}{\sqrt{D_a\tau}}\right)$$

Let's take the easiest initial plasma distribution profile.

[Top curve of Figure 5.3 with many peaks]

As we can see, the profile is symmetric, and the density goes to zero (or close to zero) on both sides of the slab. Assuming that the left and right sides of the slab is at $x = -L$ and $x = L$ respectively. Then, $S = 0$ at $x = \pm L$ (boundary of the slab). This can only happen if $B = 0$ because sine is an odd function, and cannot be zero at ± of the same value.

The cosine term, on the other hand, is an even function, and is zero when its argument is $\pm\pi/2, \pm 3\pi/2, ..., \pm(2k+1)\pi/2$, for $k = 0, 1, 2, \ldots$ then

$$\frac{L}{\sqrt{D_a \tau}} = \frac{(2k+1)\pi}{2}$$

$$\tau = \left(\frac{2L}{(2k+1)\pi}\right)^2 \frac{1}{D_a}$$

We can then write,

$$n = ST = A\cos\left(\frac{(2k+1)\pi x}{2L}\right) T_0 e^{-t/\tau}$$

At $x = 0$ and $t = 0$, n = n_0. Thus, $AT_0 = n_0$. Consequently

$$n = n_0 \cos\left(\frac{(2k+1)\pi x}{2L}\right) e^{-t/\tau}$$

The exponential term suggests that plasma decays with time.

If we take $k = 0$, the lowest order diffusion mode with one peak can be obtained where

$$n = n_0 \cos\left(\frac{\pi x}{2L}\right) e^{-t/\tau}, \quad \tau = \left(\frac{2L}{\pi}\right)^2 \frac{1}{D_a}$$

The spatial profile and time evolution result would look like this

[Figure 5.3 Chen]

Naturally, the initial profile is more chaotic than the one presented. Suppose it looks something like this

[Top curve of Figure 5.4 Chen]

Then, the density would be something like

$$n = n_0\left[\sum_k a_k e^{-t/\tau_k}\cos\left(\frac{(2k+1)\pi x}{2L}\right)+\sum_l a_l e^{-t/\tau_l}\sin\left(\frac{2l\pi x}{2L}\right)\right]$$

and

$$\tau_k = \left(\frac{2L}{(2k+1)\pi}\right)^2\frac{1}{D_a},\ \ \tau_l = \left(\frac{2L}{2l\pi}\right)^2\frac{1}{D_a}$$

where the solution contains multiple modes from combinations of k and l.

The time evolution result would be something like this

[Figure 5.4 Chen]

Diffusion in a cylindrical geometry

Here, we can write

$$n(\underline{r},t) = S(\underline{r})T(t)$$

The time part T(t) would have the same solution as in the slab case.

However in cylindrical coordinate,

$$\nabla^2 S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 S}{\partial \phi^2}+\frac{\partial^2 S}{\partial z^2}$$

If we assume that there is no change in ϕ and z direction, then the second and third terms on the right-hand-side are zero. Then,

$$\nabla^2 S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) = \frac{d^2 S}{dr^2}+\frac{1}{r}\frac{dS}{dr}$$

and consequently,

$$\frac{D_a}{S}\left[\frac{d^2S}{dr^2}+\frac{1}{r}\frac{dS}{dr}\right]=-\frac{1}{\tau}$$

$$\frac{d^2S}{dr^2}+\frac{1}{r}\frac{dS}{dr}+\frac{S}{D_a\tau}=0$$

The solution for this is a Bessel function of zeroth order.

$$S=J_0\left(\frac{r}{\sqrt{D_a\tau}}\right)$$

where it looks like this

[Figure 5.6 Chen]

Assuming that the cylindrical geometry has boundary at $r = a$. Then, we can set

$$S(a)=J_0\left(\frac{a}{\sqrt{D_a\tau}}\right)=0$$

to find τ. There are many possible values which cause J_0 to be zero. However, the first zero is what we are looking for and that corresponds to when

$$\frac{a}{\sqrt{D_a\tau}}=2.4048 \quad \text{or} \quad \tau=\left(\frac{a}{2.4048}\right)^2\frac{1}{D_a}$$

In any case, since the time part $T(t)$ is the same as before, the plasma decays.

Higher order mode can be represented by Bessel function of higher order.

Source & Sink

In the presence of source (or sink) term, i.e. term representing gain (or loss) of charged particles in plasma, then we need to modify the continuity equation to take into account the source (or sink). Then,

$$\frac{\partial n}{\partial t}-D\nabla^2 n=Q(\underline{r})$$

Positive Q indicates source term, while negative Q indicate sink term.

Ionization

This is an example of source term that can happen in plasma.

Ionization can happen when energetic electron hits neutral atom.

The process gives out more charged particles into plasma.

Here, Q is positive, and is dependent on the electron density (n) and the **ionization function** (Z):

$$Q = Zn$$

Recombination process

This is an example of sink term that can happen in plasma.

Collision between ion and electron sometimes causes them to recombine and form neutral body with certain probability.

If the resulting body is photon, then the process is called **radiative recombination**.

If the resulting body is particle, then the process is called **three-body recombination**.

Here Q is negative, and is dependent on electron and ion densities (n^2), and the **recombination coefficient** (α):

$$Q = -\alpha n^2$$

Diffusion of plasma in a magnetic field (qualitatively)

We know that

- Magnetic field dose not affect particle motion in parallel direction
- Particle only gyrate around (perpendicular to) magnetic field

[Figure from 471 Lecture under topic "diffusion of plasma in a magnetic field"]

Thus,

- if there is no collision, particle should not diffuse at all
- if there is collision, particle would diffuse across B.
    - Step length no longer depends on λ_m, but on r_L instead.
    - Thus, to slow down diffusing rate, we can decrease r_L = increase B = increase ω_c
    - As it turns out,

$$D_{\perp} = \frac{D}{1+\omega_c^2\tau^2}$$

- If $\omega_c^2\tau^2 << 1$, magnetic field has no effect on diffusion
- If $\omega_c^2\tau^2 >> 1$, magnetic field significantly reduce diffusion rate