Last Time

We have continued our discussion on diffusion in plasma.

More on diffusion in fully ionized plasma

- Bohm diffusion is the diffusion actually found in experiment as opposed to classical diffusion. The diffusion rate is much higher because of the imperfection of confinement.
    - $D_B = \frac{1}{16}\frac{KT_e}{eB}$

Diffusion in partially ionized plasma

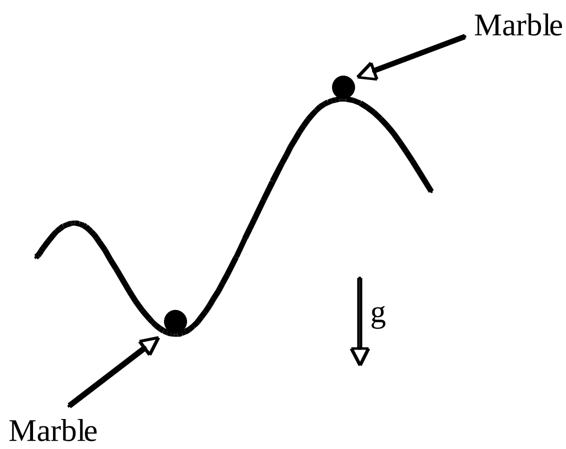
- Main collision: charged particle – neutral atom
- Ambipolar diffusion: effect of ions on electron diffusion
    - $D_a \equiv \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e}$
- Diffusion in slab geometry
- Diffusion in cylindrical geometry
- Concept of source & sink terms
- Example of source term: ionization - $Q = Zn$
- Example of sink term: recombination - $Q = -\alpha n^2$
- Plasma diffusion in magnetic field: diffuse across B in the presence of collisions.
    - If $\omega_c^2 \tau^2 << 1$, magnetic field has no effect on diffusion
    - If $\omega_c^2 \tau^2 >> 1$, magnetic field significantly reduce diffusion rate

Plasma Instability

Equilibrium vs. Stability

First, we need to understand the difference between these two states.

In a system that is in state of equilibrium, all forces acting on the system are balanced so that a system cannot be accelerated or decelerated ($\underline{F} = m\underline{a} = 0$).



The two marbles in the figure above are both in equilibrium. They can stay still where they are as long as they do not get moved out of their positions.

The marble on the left is also stable because even if it gets moved out of its equilibrium position, assuming that the movement is not too large, the marble would be restored to its original position by gravitational force.

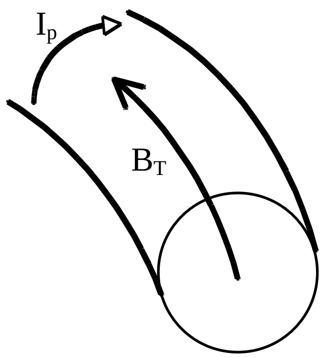
However, the marble on the right is unstable because if it gets moved out of its equilibrium position, even if the movement is small, the marble would be accelerate away from the equilibrium position by gravitational force.

Previously, I have shown an example of plasma instability in a magnetic mirror. We have seen that plasma at the central region of magnetic mirror is unstable, while plasmas at the end regions of the magnetic mirror are stable. In both cases, the plasmas are acted on by centrifugal forces, and they are both in equilibrium until perturbations are applied.

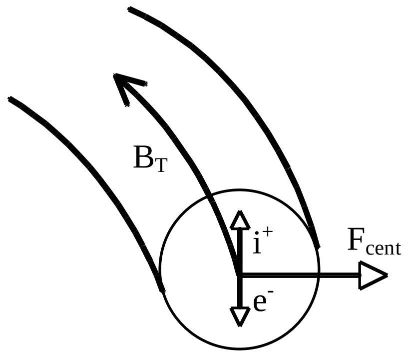
There are many types of instabilities in plasma. Here, some of the basic ones will be discussed.

Equilibrium and Stability in Tokamak Geometry (Toroidal)

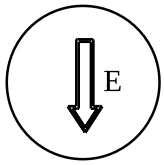
A toroidal plasma with only a toroidal magnetic field ($\underline{B}_T$) does NOT have equilibrium. Considering the following figure, polaroidal current ($\underline{I}_p$) is used to create $\underline{B}_T$.



The centrifugal force ($\underline{F}_{cent}$) due to curvature drift and $\underline{B}_T$ causes $\underline{F}_{cent} \times \underline{B}_T$ drift which separate ions and electrons. Ions move upward while electrons move downward.

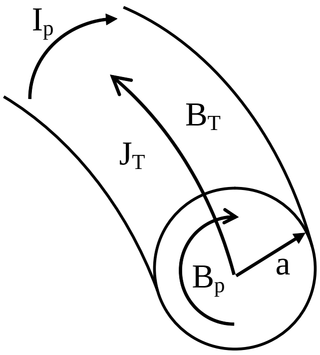


The separation results in electric field being set up between ions and electrons in the downward direction.



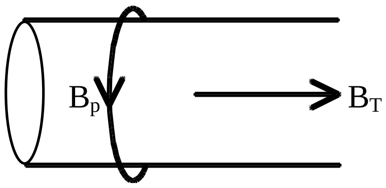
Then, there is $\underline{E} \times \underline{B}_T$ drift in the same direction as $\underline{F}$.

To provide equilibrium, toroidal current ($\underline{J}_T$) is needed to provide a polaroidal magnetic field ($\underline{B}_p$).

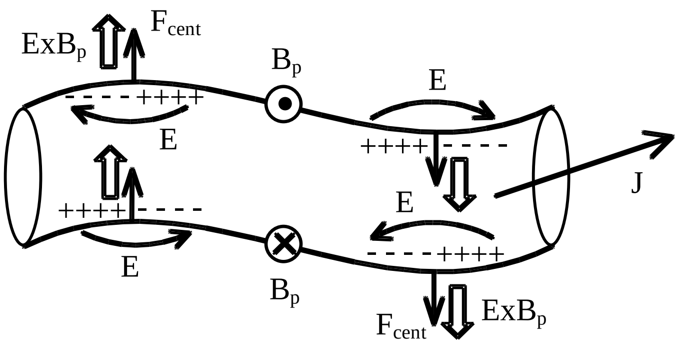


The combination of $\underline{B}_T$ and $\underline{B}_p$ provide helix field line which reduces charge separation. As an example, parameters for JET are $I_T \sim O(2\ MA)$, $B_T \sim 40\ kG$, $R \sim$ few meters, $a \sim O(1\ m)$, $T_i = 40\ keV$, $T_e = 10\ keV$, and $n = 10^{14}\ cm^{-3}$.

One of the instability that can occur in Tokamak geometry is called ***kink-mode (m=1 mode) instability***. Consider a current carrying plasma column which is part of a tokamak,



If perturbations occur and the shape of the column changes so that some part bends upward, and some part bends downward.



Let's only consider the presence of the polaroidal magnetic field for now, and look at the top left section. In this case, the plasma column bends upward. $\underline{B}_p$ is pointing out of this page, and $\underline{F}_{cent}$ is pointing upward.

- First, the $\underline{F}_{cent} \times \underline{B}_p$ drift causes ions to drift to the right, and electrons to drift to the left.
- Second, electric field is ($\underline{E}$) is set up pointing from right to left.

- Third, the $\underline{E} \times \underline{B}_p$ drift moves this section of the column upward even more.

Now let's look at the bottom left section. In this case, the plasma column also bends upward and $\underline{F}_{cent}$ is also pointing upward. However, $\underline{B}_p$ is pointing into this page.

- First, the $\underline{F}_{cent} \times \underline{B}_p$ drift causes ions to drift to the left, and electrons to drift to the right.
- Second, electric field is ($\underline{E}$) is set up pointing from left to right.
- Third, the $\underline{E} \times \underline{B}_p$ drift moves this section of the column upward even more.

Thus, the left side of this plasma column experiences upward drift which reinforces the perturbation. It is then unstable.

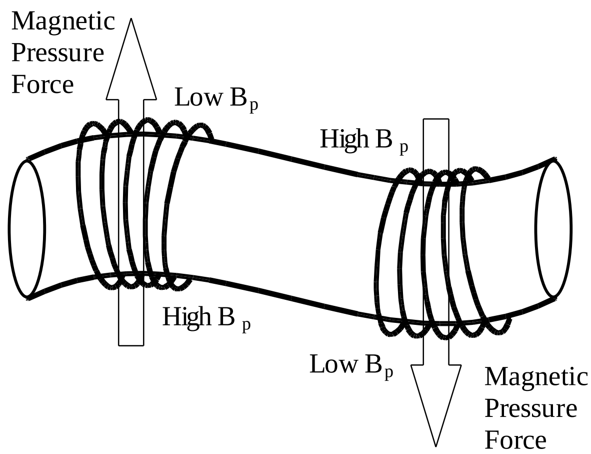
At the top right section, the plasma column bends downward. $\underline{B}_p$ is pointing out of this page and $\underline{F}_{cent}$ is pointing downward.

- First, the $\underline{F}_{cent} \times \underline{B}_p$ drift causes ions to drift to the left, and electrons to drift to the right.
- Second, electric field is ($\underline{E}$) is set up pointing from left to right.
- Third, the $\underline{E} \times \underline{B}_p$ drift moves this section of the column downward even more.

At the bottom right section, the plasma column also bends downward and $\underline{F}_{cent}$ is pointing downward as well. However, $\underline{B}_p$ is pointing into this page.

- First, the $\underline{F}_{cent} \times \underline{B}_p$ drift causes ions to drift to the right, and electrons to drift to the left.
- Second, electric field is ($\underline{E}$) is set up pointing from right to left.
- Third, the $\underline{E} \times \underline{B}_p$ drift moves this section of the column downward even more.

Thus, the right side of this plasma column experiences downward drift which reinforces the perturbation. It is then also unstable.



Another way to look at this is to notice the density gradient of $\underline{B}_p$ due to perturbation. The magnetic pressure force acts on plasma from higher $\underline{B}_p$ to lower $\underline{B}_p$ which also causes perturbation to grow. To fix this, toroidal magnetic field can be applied to help stabilize the plasma.

Ultimately, a plasma column with only polaroidal magnetic field will experience kink-mode instability, while a plasma column with only toroidal magnetic field is not in equilibrium. Balancing both magnetic fields in the tokamak geometry is therefore necessary.

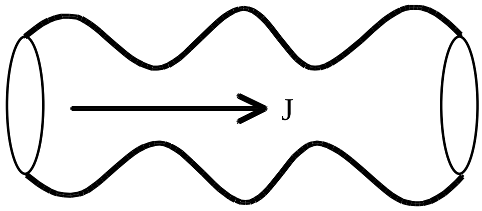
The criterion for balancing magnetic fields in a tomamak goes as

$$\boxed{\frac{B_T}{B_p} > \frac{R}{a}}$$

for stability against kink mode. This is also known as ***Kruskal Shafranov Criterion***. In typical tokamak, $B_T^2 >> B_p^2$ so that $B^2 \approx B_T^2$.

It should be noted that the plasma current ($\underline{J}$) cannot be too high because $\underline{B}_p$ will then be too high. It cannot be too low either because certain plasma density is needed.

In addition to the kink-mode instability, there is also ***sausage-mode (m=0 mode) instability***.



This instability follows the same criterion as the kink-mode instability.

It is dangerous to have one of these modes. Tokamak should never have either ones.

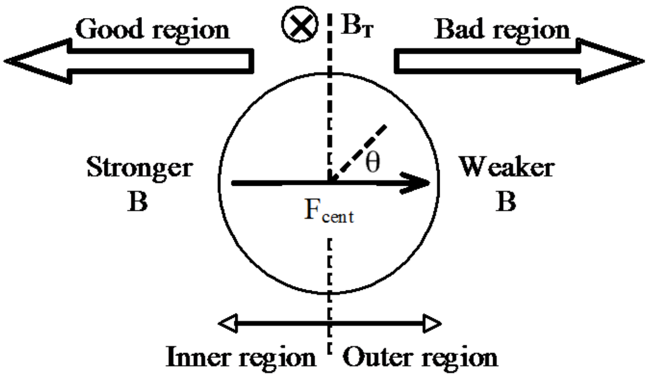
In general, amplitude of a perturbation goes as $e^{\gamma t}$, where γ is the growth rate of the perturbation. It turns out that the growth rate of the kink- and sausage- mode instabilities is

$$\gamma = kv_A$$

where

$$\begin{aligned} v_A &= \sqrt{\frac{\text{magnetic pressure}}{\text{plasma mass density}}} \\ &= \sqrt{\frac{B^2/2\mu_0}{n_0 m_i}} \end{aligned}$$

Just like magnetic mirror, tokamak has both good and bad regions in its geometry. Consider the cross section of the plasma column,



Plasma on the left side would be pushed back inside the column, while plasma on the right side would be pushed out of the column. Thus, the left side is the good region, while the right side is the bad region. Magnetic field on the left side is stronger than magnetic field on the right side because of the curvature of the toroidal.

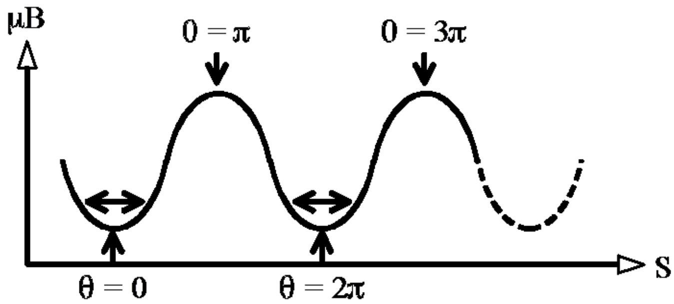
When plasma expands, then it expands to the weaker magnetic field region.

Recall that

$$\begin{aligned} E &= \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_{||}^2 \\ &= \mu B + \frac{1}{2}mv_{||}^2 \\ &= \text{constant} \end{aligned}$$

Thus, the parallel velocity ($v_{||}$) is slow in the inner region in comparison to $v_{||}$ in the outer region ($B_{inner} > B_{outer}$). Plasma, then, spend more time in the inner region where it is a good region, and less time in the outer region where it is a bad region.

Tokamak is set to have average "minimum magnetic field geometry". That is, if we look at magnetic field along the magnetic field line,



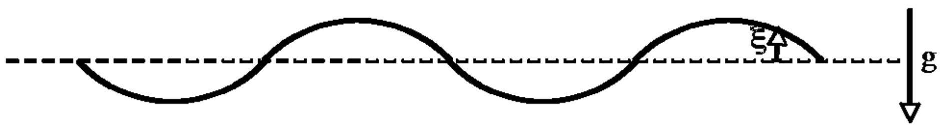
Electrons with small $v_{||}$ will be trapped. Unfortunately, they are trapped in the unfavorable regions like $\theta = 0, 2\pi, ...$ which leads to instability of confinement.

Rayleigh-Taylor Instability

The type of instability where $\underline{k} \perp \underline{B}$ like above is sometimes called ***Flute instability***, which is a subset of what is known as ***Rayleight-Taylor instability***.

R-T instability is an instability of the interface between two fluids of different densities.

Consider a standard model for R-T (Flute, gravitational, interchange, etc.) instability,



The dotted line is the interface between 2 fluids, e.g. plasma and vacuum. We can write,

$$\frac{d^2\xi}{dt^2} = (\ ?\ )\xi$$

Let $\xi = Ae^{-i\omega t}$, then $\omega^2 = -(\ ?\ )$. So, what is $(\ ?\ )$?

- First, it must contain the term *g* which represents force.
- Second, if it is linearly proportional to *g* then, the other term must have [1/length] unit.
- Without going into detail (can be found in Chen's), it is found that

$$\omega^2 = -gk$$

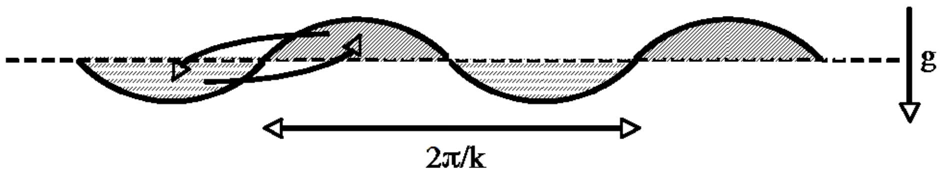
  where k is the wave number which depends on a system that we look at.

Consequently, $\omega = \pm i\sqrt{gk}$, and $\xi = Ae^{-i\left(\pm i\sqrt{gk}\right)t} = Ae^{\pm\sqrt{gk}t}$ and $= Ae^{\sqrt{gk}t}$ if we are looking at growth term. As before, we can define γ to be growth rate. Thus,

$$\gamma = \sqrt{gk}$$

Consider ordinary fluid,

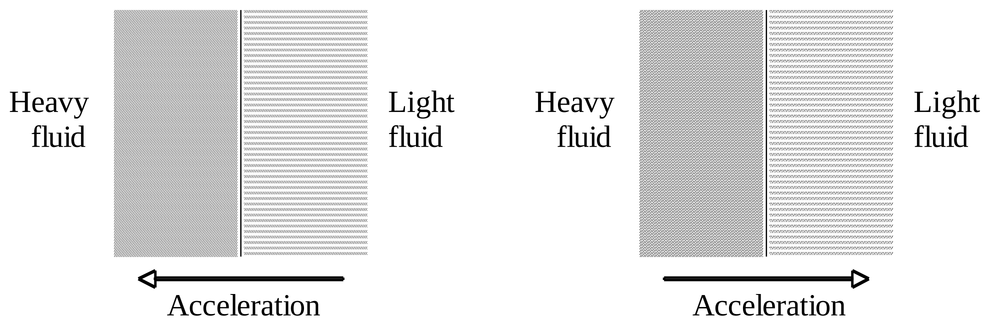
It is easy to see that this geometry is unstable. What happen then at the interface?



Light water exchanges place with heavy water. The perturbation at the interface will grow, and eventually, all of the heavy water will change place with the light water.

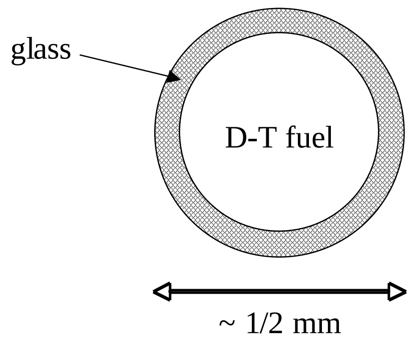
This geometry is stable.

Suppose we now have the following geometries instead,



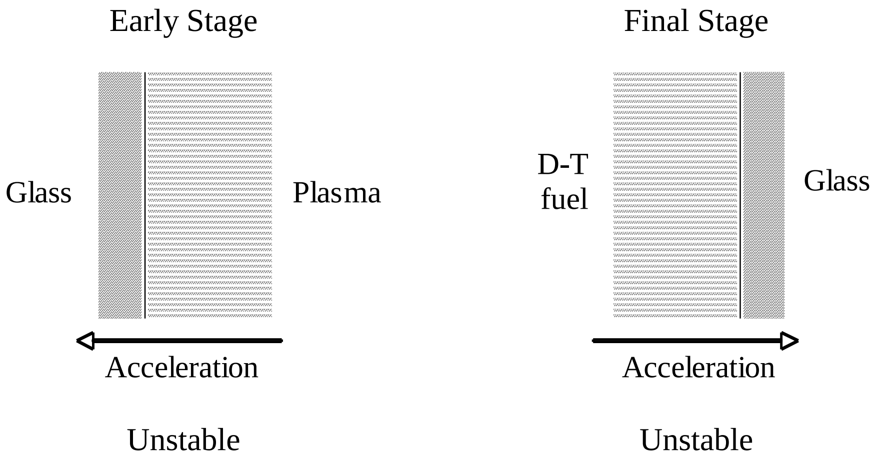
Here, the left geometry is an unstable one, while the right geometry is a stable one. The reason that this is a little different from the previous case is that while the system accelerates in one direction, the force experienced by fluid is in the opposite direction.

Another example of R-T instability is in inertial confinement fusion. Consider a D-T capsule target for inertial confinement



What happen in the process is as followed:

- First, photon (energetic beam) comes in.
- Plasma is injected out.
- Capsule implodes and compresses the D-T fuel. (Pressure after compression is about 100 Mbar due to high temperature)
- The outer surface of capsule suffers R-T instability at early stage (of inward acceleration) of implosion.
- The inner surface of capsule suffers R-T instability at final state (radial deceleration) of implosion.



The reaction force of the force due to ejection of the plasma accelerates the glass inward.

Glass moves inward, but decelerates. The D-T fuel is lighter than glass.

Because of these instability, the impurity can get into the glass, resulting in asymmetry in the compression, which is bad for inertial confinement.

Two-stream instability

This kind of instability occurs when there are fluids with more than one zeroth order velocities. In this case, we limit it to two fluids.

The two fluids can be of the same or different species. For instance, they can both be electron streams, or one can be an electron stream while the other is an ion stream.

Let's consider the latter situation, and let the ion stream's zeroth order velocity be zero, and the electron stream's zeroth order velocity be V_0.

Recall the dielectric function,

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{(\partial g_e(v)/\partial v)}{(v-\omega/k)}dv-\frac{\omega_{pi}^2}{k^2}\int_{-\infty}^{+\infty}\frac{(\partial g_i(v)/\partial v)}{(v-\omega/k)}dv$$

Then, the dispersion relation can be readily derived to be

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{(\omega-kV_0)^2}-\frac{\omega_{pi}^2}{\omega^2}=0$$

This is a quartic equation with 4 solutions of ω. Since this is a real equation, a complex conjugate of any root will also be a root.

Notice that our wave quantities go as $\sim e^{-i\omega t}$. Then,

- If ω is all real, then they would only be oscillating.
- If ω has complex roots, they would come in pair, and the wave quantities would then go as $\sim e^{\pm \mathrm{Im}[\omega]t}$. One of the roots makes the wave quantities grow, while its complex conjugate makes them decay.

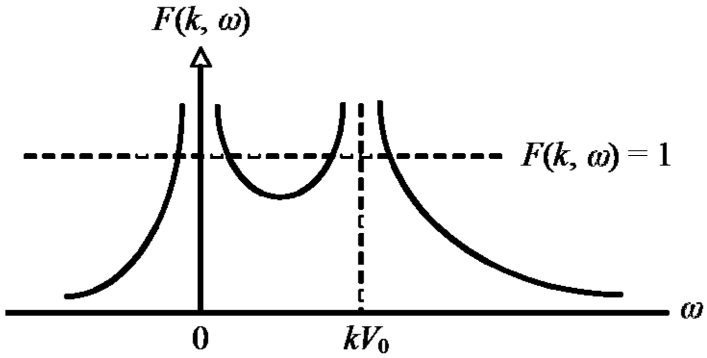
Thus, if we can find that ω has even one complex root, we can conclude that instability occurs. This instability is known as **two-stream instability**.

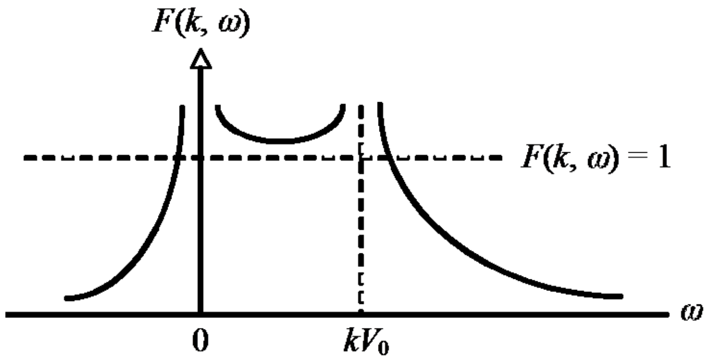
To do that, we shall define

$$F(k,\omega)\equiv\frac{\omega_{pe}^2}{(\omega-kV_0)^2}+\frac{\omega_{pi}^2}{\omega^2}$$

and plot this function $F(k,\omega)$ against real value of ω. We shall then draw horizontal line at $F(k,\omega)=1$.

- If the horizontal line cuts 4 points of the curve, then we shall conclude that all 4 roots of ω are real.
- If, however, the horizontal line only cuts 2 points of the curve, then the other 2 roots must be complex number, and thus there will be instability.





To find the condition where instability exists, we can find the solution when

$$F_{\min} > 1$$

where $F_{\min}$ is determined by $\partial F/\partial \omega = 0$. [See Homework]

An interesting concept which we shall not go into detail derivation here is that of **negative energy**. Electron oscillation has negative energy. What this mean is that the oscillation grows when energy is taken out of electron. Ion oscillation, however, has positive energy, so it grows when energy is given to it.

In the two-stream instability case, wave takes out energy from electron stream and dumps it on ion stream. Consequently, both electron and ion wave grow, while the total energy is kept constant.

Two-stream instability is quite common in plasma physics. It suggests that Maxwellian distribution of plasma is desirable to avoid this instability. That is, we do not want to have double peak distribution.