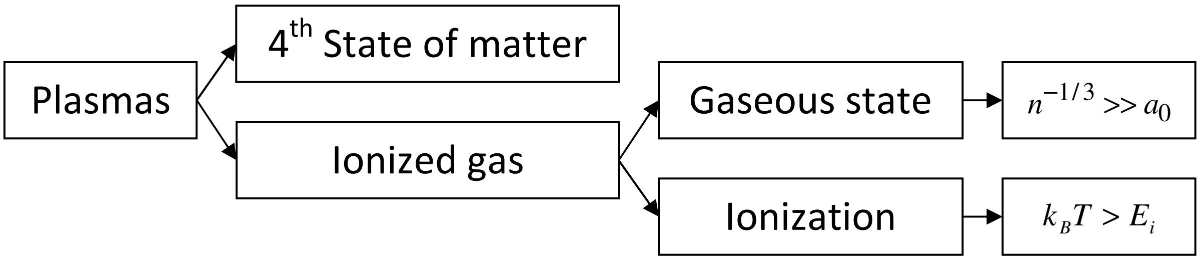
Last Time



Plasmas in nature and man-made plasmas

Plasma temperature

- Plasma can have more than one temperature
- Plasma temperature relates to velocity, hence energy, of charged particles
- Plasma with temperature = KT has average energy = ½KT per degree of freedom

Basic Plasma Properties

We know that plasma is ionized gas, but there's always some degree of ionization in gas. So, what kind of basic properties can we expect to see from gas that has been ionized to be called plasma?

From last time, we have this equation

$$k_B T > E_i\text{, } n^{-1/3} >> a_0\text{, and } E_i \cong \frac{e^2}{4\pi\varepsilon_0 a_0}$$

So,

$$n^{-1/3} >> \frac{e^2}{4\pi\varepsilon_0 E_i} >> \frac{e^2}{4\pi\varepsilon_0 k_B T}.$$

We can do some further manipulation

$$n^{-1/3}\frac{4\pi\varepsilon_0 k_B T}{e^2} >> 1$$

$$4\pi n^{2/3}\left(\frac{\varepsilon_0 k_B T}{e^2 n}\right) >> 1$$

$$(4\pi)^{3/2} n\left(\frac{\varepsilon_0 k_B T}{e^2 n}\right)^{3/2} >> 1.$$

Since $(4\pi)^{3/2} \approx 45$, we can omit it. Then,

$$\boxed{n\lambda_D^3 >> 1}$$

where

$$\boxed{\lambda_D = \left(\frac{\varepsilon_0 k_B T}{e^2 n}\right)^{1/2} \propto \left(\frac{T}{n}\right)^{1/2}}$$

This λ_D is an important length scale of plasma, and is called **Debye length**. What this tells us is that in a cubic or spherical volume of about λ_D^3, there is large number of gas particles inside. The $n\lambda_D^3$ is sometimes called **plasma parameter Λ**.

For JET tokamak, n is about 10^{14} cm^{-3} and k_BT is about 10 keV. So λ_D is around 7.4E-3 cm.

$n\lambda_D^3$ is bout 4.05E+7.

For interstellar medium, n is about 1 cm^{-3} and k_BT is about 1 eV. So λ_D is around 740 cm.

$n\lambda_D^3$ is about 4.05E+8.

So what is this λ_D physically?

The electron and ion distributions inside plasma in the presence of potential ϕ are

$$n_e = n_{e0} \exp\left(e\phi / k_B T_e\right) \text{ and } n_i = n_{i0} \exp\left(-e\phi / k_B T_i\right)$$

so that when ϕ is 0 and $n_{e0} = n_{i0}$ (neutral plasma), $n_e = n_i$. This is called Boltzmann relation and we will discuss about it later.

Suppose we want to insert a point charge q_T into the plasma, the potential ϕ will follow the **Poisson's equation**,

$$\boxed{\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}}$$

Here $\rho = e[n_i - n_e] + q_T \delta(\vec{r})$ where $\delta(\vec{r})$ is the unit delta function indicating location of the point charge. So,

$$\varepsilon_0 \nabla^2 \phi = -e[n_i - n_e] - q_T \delta(\bar{r})$$

$$\varepsilon_0 \nabla^2 \phi + e\left[n_{i0} \exp\left(-e\phi/_{k_B T_i}\right) - n_{e0} \exp\left(e\phi/_{k_B T_e}\right)\right] = -q_T \delta(\bar{r}).$$

Let $n_{e0} = n_{i0} = n_0$ and Taylor's expand the exponential terms, then

$$\varepsilon_0 \nabla^2 \phi + en_0\left[\frac{e\phi}{k_B T_i} - \frac{e\phi}{k_B T_e}\right] = -q_T \delta(\bar{r})$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\right] - \frac{\phi}{\lambda_D^2} = -\frac{q_T}{\varepsilon_0}\delta(\bar{r})$$

where we have written the derivative in spherical coordinate, and let $1/\lambda_D^2 = 1/\lambda_{Di}^2 + 1/\lambda_{De}^2$. We only need to look at the change in r direction because there is uniformity in the θ and ϕ directions.

Let put the point charge at r = 0, so

For r > 0, $$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\right] - \frac{\phi}{\lambda_D^2} = 0$$

Let $\phi = F(r)/r$, then

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\left(\frac{F(r)}{r}\right)\right] - \frac{1}{\lambda_D^2}\left(\frac{F(r)}{r}\right) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{r^2}\right)\frac{dF(r)}{dr}\right] - \frac{1}{\lambda_D^2}\left(\frac{F(r)}{r}\right) = 0$$

$$-\frac{1}{r^2}\frac{d^2F(r)}{dr^2} - \frac{1}{\lambda_D^2}\left(\frac{F(r)}{r}\right) = 0$$

$$F(r) = A\exp(-r/\lambda_D) + B\exp(r/\lambda_D).$$

But F(r) must approach zero as r approaches infinite for the solution to exist, so we can reject the second term. Then

$$\phi(r) = \frac{A}{r}\exp\left(-\frac{r}{\lambda_D}\right).$$

As r approaches zero, $\phi = q_T / r$ as if there is no plasma, so

$$\phi(r) = \frac{q_T}{r} \exp\left(-\frac{r}{\lambda_D}\right).$$

Notice that the effect of plasma comes in the exponential term. So, what this equation says is that if we put a charge inside plasma, plasma would shield out the potential so that at distance approximately λ_D further away, plasma will not notice the presence of the charge at all. This behavior is also called **shielding cloud**, and the spherical volume surrounding is called **Debye sphere**. Any electrical potential applied from outside to plasma would be shielded out. Two charges which are outside Debye sphere of each other cannot see one another. This is also why plasma has "dielectric" property, which is a property of certain materials that decrease electric field.

Once again, consider this equation

$$n^{-1/3} >> \frac{e^2}{4\pi\varepsilon_0 k_B T}.$$

It can also be written as

$$\frac{k_B T}{e^2 / 4\pi\varepsilon_0 n^{-1/3}} = \frac{K.E.}{P.E.} >> 1.$$

That is, the particle kinetic energy is much greater than the average Coulombic interaction energy.

Because of this, plasma can be treated as continuum or collection of particles rather than individual particle.

To see this in a clearer picture, we can consider collision process in plasma.

Suppose we send in a test particle into plasma (again) which has density n. The distance that the test particle would travel before it collides with another particle is defined as

$\lambda = 1/n\sigma$ = mean free path

where σ = cross section of the collision event. So

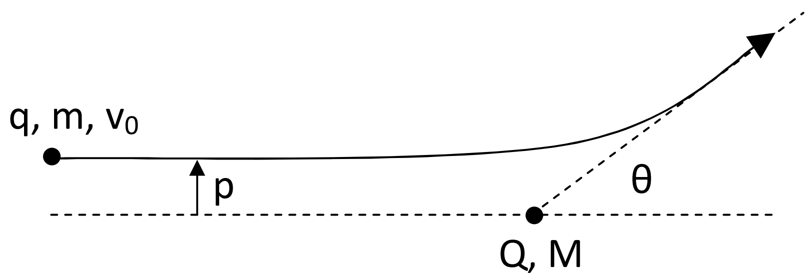
collision time = λ/v = 1/(nσv) = τ

collision frequency = 1/τ = nσv = ν

where v = velocity of the test particle.

Now we need to calculate σ.

Assume that the test particle has charge q, mass m, and is moving toward the target particle at velocity v_0. The target particle has charge Q and mass M. Let's assume for now also that the charges q and Q have the same sign so they repel each other.



p is called impact parameter where we define that if the particle is coming in at distance

> p, then the scattering angle θ would be smaller than 90 degree = small angle scattering, and

< p, then the scattering angle θ would be larger than 90 degree = large angle scattering.

It can be proven that at closest separation between q and Q, K.E. of the test particle = P.E. of the test particle and θ = 90 degree. (Proof = Homework exercise) Let's define the closet distance between the test and the target particle as p_0, i.e. distance of closest approach. Thus,

$$\frac{1}{2}mv_0^2 = \frac{qQ}{4\pi\varepsilon_0 p_0}$$

$$p_0 = \frac{qQ}{2\pi\varepsilon_0 mv_0^2}$$

So the cross section for large angle scattering is then

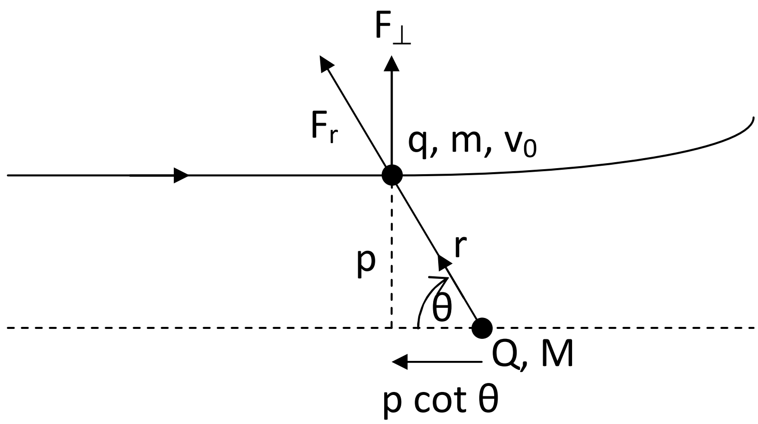
$$\sigma_L = \pi p_0^2 = \pi\left(\frac{qQ}{2\pi\varepsilon_0 mv_0^2}\right)^2,$$

and the collision frequency for the large angle scattering is

$$\boxed{\nu_L = nv_0\sigma_L = \frac{nq^2Q^2}{4\pi\varepsilon_0^2 m^2 v_0^3}}.$$

For small angle scattering, remember that beyond the Debye length, charges cannot see each other. So, the distance p lies between the

$$\frac{qQ}{2\pi\varepsilon_0 m v_0^2} < p \le \lambda_D \,.$$



As the test particle move past the target particle, the perpendicular velocity $v_\perp$ varies. The change in momentum, i.e. net impulse, can be calculated from the integration of the Coulomb's force in perpendicular direction,

$$\Delta\left(mv_\perp\right) = \int_{-\infty}^{\infty} F_\perp dt \,,$$

where

$$\begin{aligned} F_\perp &= F_r \sin\theta \\ &= \frac{qQ}{4\pi\varepsilon_0 r^2}\sin\theta \end{aligned}$$

Since $p = r\sin\theta$,

$$F_\perp = \frac{qQ}{4\pi\varepsilon_0 p^2}\sin^3\theta \,.$$

Also,

$$\begin{aligned} v_0 &= -\frac{d}{dt}\left(p\cot\theta\right) \\ &\approx -p\frac{d}{dt}\left(\cot\theta\right) \approx \frac{p}{\sin^2\theta}\frac{d\theta}{dt} \end{aligned}$$

so

$$dt \approx \frac{p}{v_0 \sin^2\theta} d\theta .$$

Then,

$$\Delta(mv_\perp) = \int_0^\pi \frac{qQ}{4\pi\varepsilon_0 p^2} \sin^3\theta \frac{p}{v_0 \sin^2\theta} d\theta$$
$$= \frac{qQ}{2\pi\varepsilon_0 p v_0}$$

Suppose there are N particles, but they are statistically uncorrelated. Then,

$$\left\langle (\Delta(mv_\perp))^2_{total} \right\rangle = \left\langle (\Delta(mv_\perp))^2_1 \right\rangle + \left\langle (\Delta(mv_\perp))^2_2 \right\rangle + \ldots + \left\langle (\Delta(mv_\perp))^2_N \right\rangle$$
$$= N\left\langle (\Delta(mv_\perp))^2 \right\rangle$$

and

$$N\left\langle (\Delta v_\perp)^2 \right\rangle = \frac{Nq^2Q^2}{4\pi^2\varepsilon_0^2 m^2 p^2 v_0^2} .$$

For cylindrical volume of length L between radius p and dp, the total number of test particles is $n(2\pi L p dp)$. Then for small angle scattering where $\frac{qQ}{2\pi\varepsilon_0 m v_0^2} < p \le \lambda_D$,

$$d\left\langle (\Delta v_\perp)^2 \right\rangle = \int_{\left(\frac{qQ}{2\pi\varepsilon_0 m v_0^2}\right)}^{\lambda_D} \left(\frac{n 2\pi L p q^2 Q^2}{4\pi^2 \varepsilon_0^2 m^2 p^2 v_0^2} \right) dp$$
$$= \frac{nLq^2Q^2}{2\pi\varepsilon_0^2 m^2 v_0^2} \left[\ln(\lambda_D) - \ln\left(\frac{qQ}{2\pi\varepsilon_0 m v_0^2} \right) \right]$$
$$= \frac{nLq^2Q^2}{2\pi\varepsilon_0^2 m^2 v_0^2} \ln\left(\frac{2\pi\varepsilon_0 m v_0^2 \lambda_D}{qQ} \right)$$

Not that if we talk both charges are proton (or electron), then q = Q = e (or –e), and

$$\frac{2\pi\varepsilon_0 m v_0^2 \lambda_D}{qQ} = \frac{2\pi\varepsilon_0 m v_0^2 \lambda_D}{e^2} = \frac{2\pi\varepsilon_0 k_B T \lambda_D}{e^2} = 2\pi n \lambda_D^3 .$$

Also, $L = v_0 dt$, so

$$\frac{d}{dt}\frac{\left\langle (\Delta v_\perp)^2 \right\rangle}{v_0^2} = \frac{nq^2Q^2}{2\pi\varepsilon_0^2 m^2 v_0^3}\ln\left(2\pi n\lambda_D^3\right) = \nu_L \times 2\ln\left(2\pi n\lambda_D^3\right).$$

Scattering time due to small angle collision is conveniently defined as time it takes for $\left\langle (\Delta v_\perp)^2 \right\rangle$ to be the same as v_0^2. Thus, the small angle collision frequency is

$$\boxed{\nu_S = \nu_L \times 2\ln\left(2\pi n\lambda_D^3\right)}.$$

The term $2\ln\left(2\pi n\lambda_D^3\right)$ does not change much but is greater than 1 (~ 10's). So collisions in plasma are predominantly small angle collisions, and the Coulombic interaction between 2 particles are not as important as interaction as a collection of particles.

Remarks:

1. Notice that

$$\nu_{Collision} = \nu_L + \nu_S \approx \nu_S \propto \frac{n}{T^{3/2}}.$$

   As temperature goes up, collision frequency goes down resulting in less loss which is good for plasma.

2. Since $m^2 v_0^3 \propto m^{1/2} T^{3/2}$, collision frequency depending only on mass and temperature of the test particle, and not the target particle. So,

$$\nu_{ee} \sim \nu_{ei}$$

$$\nu_{ii} = \sqrt{\frac{m_e}{m_i}}\nu_{ee}$$

$$\nu_{ie} = \frac{m_e}{m_i}\nu_{ee}$$

   Factor of $\sqrt{m_e/m_i}$ is introduced from center-of-mass calculation.

3. For deuterium plasma in tokamak, n = 10^{14} cm^{-3}, $T_e \approx T_i$ = 10 keV. Then,

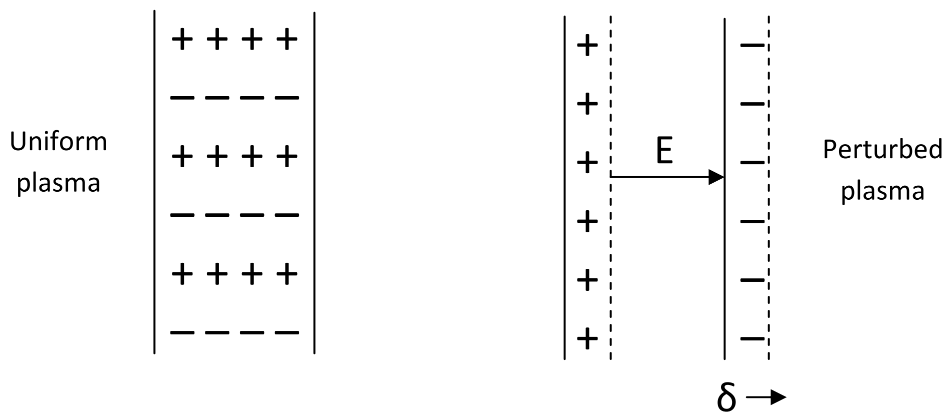
$\nu_{ee} = \nu_{ei} \approx 5.7 \times 10^3$ Hz, $\nu_{ii} \approx 95$ Hz, and $\nu_{ie} \approx 1.6$ Hz or

$\tau_{ee} = \tau_{ei} \approx 0.18$ ms, $\tau_{ii} \approx 11$ ms, and $\nu_{ie} \approx 0.6$ s.

Just for scaling, time scale for D-D fusion reaction in this type of tokamak is about 4 hours.

For D-T, it's about 140 sec.

Plasma is in fact quasi-neutral. The shielding cloud will try to restore neutrality to plasma when it is perturbed (i.e. by small external electric field) by moving the charge cloud around to adjust the charge distribution. When this happen, it tends to overshoot and oscillate around the equilibrium position.



Let σ = surface charge density = $-en_0\delta$.

So $E = \sigma/\varepsilon_0 = -en_0\delta/\varepsilon_0$.

$$m_e\ddot{\delta} = eE = -\frac{e^2 n_0}{\varepsilon_0}\delta$$

$$\ddot{\delta} + \left(\frac{e^2 n_0}{\varepsilon_0 m_e}\right)\delta = 0$$

This is a common form of oscillating system which has oscillation frequency of

$$\omega_{pe} \equiv \left(\frac{e^2 n_0}{\varepsilon_0 m_e}\right)^{1/2} \text{ = plasma frequency.}$$

In this case, it is electron plasma frequency. Usually, ion is much slower to move due to its large mass, so electron is the charge particle that does the work. Plasma frequency depends on n, and is usually very high for lab plasma (in the microwave range)

For processing plasma, $n = 10^{12}$ cm^{-3}, ω_{pe} = 56.5E+9 rad/s or 9 GHz.

For tokamak, $n = 10^{14}$ cm^{-3}, ω_{pe} = 565E+9 rad/s or 90 GHz. This is time scale for collective interaction and is much larger than the collision time which is time scale for individual interaction!