Last time

- Debye's length $\lambda_D = \left(\frac{\varepsilon_0 k_B T}{e^2 n}\right)^{1/2} \propto \left(\frac{T}{n}\right)^{1/2}$
    - Debye sphere
    - Debye shielding
- Plasma parameter $\Lambda = n\lambda_D^3 >> 1$
- Collision frequency $\nu = n\sigma v$, collision time $\tau = 1/\nu$
- Small/large angle Coulomb collision
    - $\nu_L = n v_0 \sigma_L = \frac{nq^2Q^2}{4\pi\varepsilon_0^2 m^2 v_0^3}$
    - $\nu_S = \nu_L \times 2\ln\left(2\pi n\lambda_D^3\right)$
    - Small angle collision dominates in plasma
- Plasma frequency $\omega_{pe} \equiv \left(\frac{e^2 n_0}{\varepsilon_0 m_e}\right)^{1/2}$

Fusion

Fusion is another nuclear process that gives out net energy (i.e. exothermic, Q > 0). As oppose to fission which involves breaking up of very heavy nuclei like uranium, fusion involves combining of very light nuclei like deuterium (D) to form a nucleus.

How do we know that fusion reaction is possible?

- We can look at the sun! The sun is a self-sustaining thermonuclear reactor whose output has been stable for the past 1 billion years.

Advantages of fusion over fission as energy source:

- Light nuclei are abundance. 0.015% of natural hydrogen is deuterium (2H1) which is one of the principle atoms commonly used to produce fusion reaction. Sound very little, but consider that a cubic meter of sea water contains 30 grams of deuterium. One gallon of sea water is equivalent to about 300 gallon of gasoline for energy production. This makes it the most abundance energy source in comparison to others.

- The products of fusion reactions are usually light, stable nuclei. So there is no need to worry about radioactive decay as oppose to the product from fission reactions.

Disadvantages

- It is very difficult to make atoms combine each other due to Coulomb repulsion force which inversely vary with the square of the distance between atoms.

Examples of fusion reactions:

- Solar fusion (Proton-Proton fusion reaction)

$$
\begin{aligned}
&{}_1^1H + {}_1^1H \rightarrow {}_1^2D + e^+ + \nu = 1.44\ MeV \\
&{}_1^2D + {}_1^1H \rightarrow {}_2^3He + \gamma = 5.49\ MeV \\
&{}_2^3He + {}_2^3He \rightarrow {}_2^4He + 2{}_1^1H + \gamma = 12.86\ MeV \\
&\Rightarrow 26.7\ MeV\ \ per\ 4{}_1^1H = 6.7\ MeV\ \ per\ {}_1^1H
\end{aligned}
$$

The reaction rate of the first reaction is very small in the order of 10^{-33} b. However, the proton is very abundance on the sun (of the order of 10^{56}), which enable the proton-proton reactions

- Deuterium-Deuterium fusion reaction

$$
\begin{aligned}
&{}_1^2D + {}_1^2D \rightarrow {}_1^1H(3\ MeV) + {}_1^3T(1\ MeV) = 4\ MeV \\
&{}_1^2D + {}_1^2D \rightarrow {}_0^1n(2.5\ MeV) + {}_2^3He(0.8\ MeV) = 3.3\ MeV \\
&{}_1^2D + {}_1^3T \rightarrow {}_0^1n(14.1\ MeV) + {}_2^4He(3.5\ MeV) = 17.6\ MeV \\
&{}_1^2D + {}_2^3He \rightarrow {}_2^4He(3.6\ MeV) + {}_1^1H(14.7\ MeV) = 18.3\ MeV \\
&\Rightarrow 43.2\ MeV\ \ per\ 6{}_1^2D = 7.2\ MeV\ \ per\ {}_1^2D
\end{aligned}
$$

This reaction is what we want to go for ultimately, but it requires much higher energy to make than the following reaction which is currently achievable on earth.

- Deuterium-Tritium fusion reaction

$$
{}_1^2D + {}_1^3T \rightarrow {}_0^1n(14.1\ MeV) + {}_2^4He(3.5\ MeV) = 17.6\ MeV
$$

This reaction is currently possible in laboratory on earth. However, tritium is not a natural occurring element (can be breed from D-D reaction), and most of the energy produces is in neutron which is hard to extract.

Each of these reactions has different cross section and rate. There are many other reactions but most of them are unlikely or happen very rarely.

Why light nuclei?

First let us look at the concept of **binding energy**.

Binding energy *B* of a nuclide ${}_Z^A X_N$ is the difference between its atomic mass energy and the total mass energy of its constituents. So,

$$B\left({}_Z^A X_N\right) = \left\{Zm_p + Nm_n + Zm_e - m\left({}_Z^A X_N\right)\right\}c^2$$

In a reaction, says M + N = O + P, the total number of protons, neutrons, and electrons is conserved. However, the atomic mass energy is not. Some mass is converted into energy, or some energy is absorbed. We can see that

$$m(M) + m(N) - m(O) - m(P) = B(O) + B(P) - B(M) - B(N) \equiv Q$$

We can define the difference in mass energy as Q. To have exothermic reaction, Q must be positive because energy is released, i.e. Q = fusion energy release.

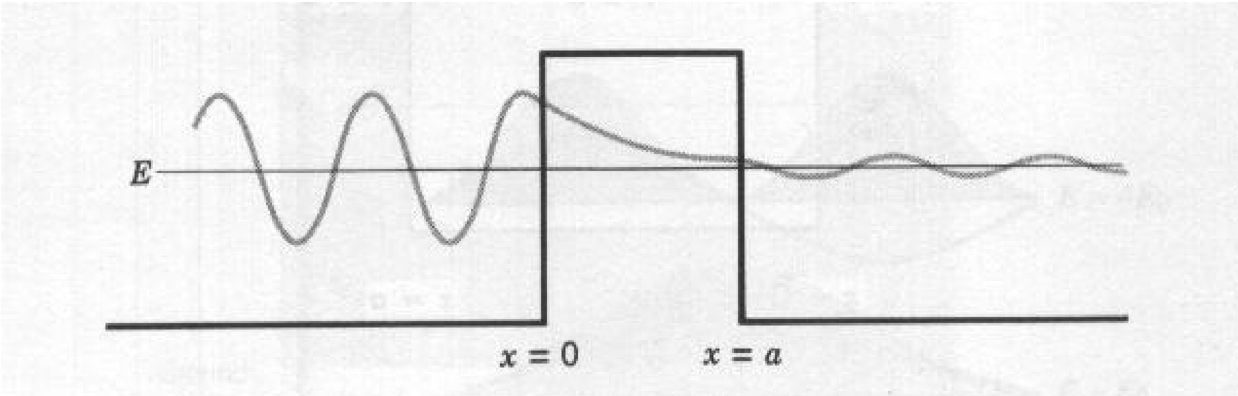
Figure 3.16 from Krane shows the binding energy of per nucleon as a function of mass number A. We can see that if we combine nuclei with low mass number (light nuclei), the binding energy of the resulting nuclei would be greater than the starting nuclei. So Q would be positive. However, if we combine nuclei with high mass number (heavy nuclei), the binding energy of the resulting nuclei would be lower, and Q would be negative, resulting in loss of net energy. This is one of the reasons why fusion reaction occurs between light nuclei.

Coulomb Barrier

When nuclei get close to each other, they experience repulsion force which prevents them from fusing with one another. This repulsion force can be viewed in a form of energy barrier,

$$V_C = \frac{e^2}{4\pi\varepsilon_0} \frac{Z_a Z_b}{R_a + R_b}$$

where R_a and R_b are respectively the radii of nuclei a and b. As Z_a and Z_b get larger, the barrier becomes stronger and is harder to fuse. So, this is another reason why fusion reaction occurs between light nuclei. Among the fusion reactions shown earlier, barrier of the D-T reaction is the lowest (around 0.4 MeV). This is still lower than the typical particle energy achievable which is around 10 keV. So how can we achieve D-T fusion? The answer is the **tunneling effect** which allows barrier penetration in term of probability (draw tunneling effect).



Without proving (since this is quantum problem), we can write the **cross section** of fusion reaction as

$$\sigma \propto \frac{1}{v^2} e^{-2G}$$

where $G = \frac{e^2}{4\pi\varepsilon_0} \frac{\pi Z_a Z_b}{\hbar v}$ for particle with relative velocity v. This relation is incomplete as it only includes energy-dependence factor. (See figure 14.1) Notice in figure 14.1 that D-T has higher cross section than D-D, but according to the equation above, it shouldn't matter as long as particles have the same Z. What are not included in this equation are the nuclear matrix elements and the statistical factor which depends on the spins of the particles.

The factor that we are more interested in is the **fusion reaction rate**. From last lecture, this reaction rate or ν = nσv. We also learn that the velocity distribution in plasma is like

$$n(v) \propto \exp\left(-\frac{1}{2} m v^2 \Big/ k_B T \right)$$

where $n(v)v^2dv$ = relative probability to find a particle with speed between v and v + dv.

Thus average σv over all velocities is

$$\langle \sigma v \rangle \propto \int_0^\infty \frac{1}{v} e^{-2G} e^{-\frac{1}{2} m v^2 \big/ k_B T} v^2 dv = \int_0^\infty e^{-2G} e^{-E/k_B T} dE \,.$$

The result is shown in Figure 14.3. As we can see, D-T reaction has the highest reaction rate, and the maximum is between 10-100 keV.

The energy release per unit volume from, for example, D-T fusion reaction is then

$$E_f = \frac{1}{4} n^2 \langle \sigma v \rangle Q \tau \,.$$

Here we assume that the density of D and T are each n/2 so that total density of electrons (and ion) is n. Q_T = energy released per reaction (17.6 MeV) and τ is the confinement time.

Let's n = 10^{14} cm-3, Q = 17.6 MeV, and T = 10 keV so that 10^{-16} cm^3/s. Then

$$\begin{aligned}\frac{E_f}{\tau} &= \frac{1}{4} \times \left(10^{14}\ cm^{-3}\right)^2 \times \left(10^{-16}\ cm^3 / s\right) \times \left(17.6 \times 10^6\ eV\right) \times \left(1.6 \times 10^{-19}\ J / eV\right) \\ &= 0.704\ W / cm^3\end{aligned}$$

To be able to get fusion energy, E_f needs to be greater than the thermal energy, which is

$$E_{thermal} = \frac{3}{2} n_e k_B T_e + \frac{3}{2} n_i k_B T_i = 3 n k_B T$$

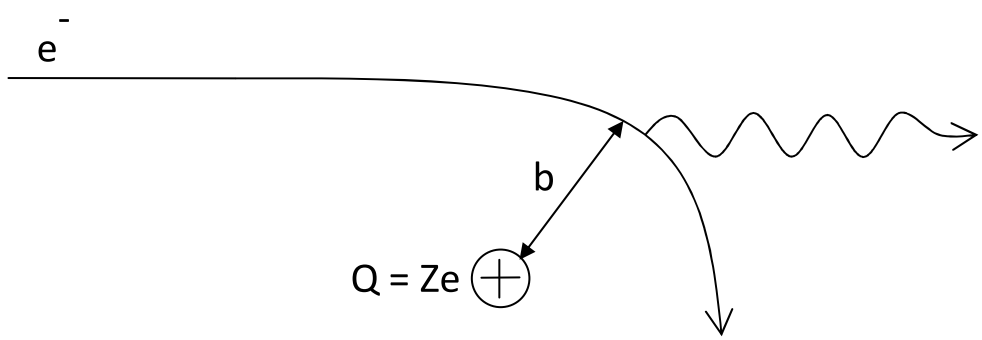
If n = n_e = n_i and T = T_e = T_i. So we have that

$$\frac{1}{4} n^2 \langle \sigma v \rangle Q \tau > 3 n k_B T$$

$$\boxed{n \tau > \frac{12 k_B T}{\langle \sigma v \rangle Q}}$$

This is known as Lawson's Criterion, which tells us the operating condition for fusion reactor, i.e. to heat plasma of density n to temperature T for time τ. For JET tokamak, $k_B T$ = 10 keV, <σv> = 10^{-22} m^3/s, nτ > 10^{20} s/m^3.

Unfortunately, fusion is not the only reaction occurring. There are other **competing reactions** which cause loss of power. The most important one is **Bremsstrahlung radiation production**. This is the result of Coulomb scattering between 2 particles, which causes to deflect and decelerate, and in turn emit radiation. Since electron is the lightest particle, it experiences the largest deceleration (draw Bremsstrahlung radiation). However, in plasma where electron and ions are in thermal equilibrium, loss by electron is felt by ions as well.



The power radiated by electron being accelerated by acceleration $\dot{v}$ is

$$P = \frac{e^2 |\dot{v}|^2}{6\pi\varepsilon_0 c^3}.$$

In this case,

$$F = m_e \dot{v} = \frac{Ze^2}{4\pi\varepsilon_0 b^2} \Rightarrow \dot{v} = \frac{Ze^2}{4\pi\varepsilon_0 m_e b^2}.$$

Thus,

$$P = \frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{Ze^2}{4\pi\varepsilon_0 m_e b^2}\right)^2 = \frac{2e^6 Z^2}{3(4\pi\varepsilon_0)^3 c^3 m_e^2 b^4}$$

If we consider electrons inside a ring of radius r (Figure 14.6 Krane), then the total number of electrons is $n_e L 2\pi r dr$. So (change b to r)

$$dP = \frac{2e^6 Z^2}{3(4\pi\varepsilon_0)^3 c^3 m_e^2 r^4} n_e L 2\pi r dr,$$

where L = $v_e$dt. If the interaction time dt is approximated to be about r/v_e (sometimes 2r/v_e), then L = r and

$$dP = \frac{4\pi e^6 Z^2 n_e}{3(4\pi\varepsilon_0)^3 c^3 m_e^2 r^2} dr$$

The value of r goes from r_{min} to r_{max}. We can take r_{max} = infinity. We cannot take the Debye's length as r_{max} because of the time scale for the electron is much shorter than the time scale for ion. r_{min} is determined by Heizenberg uncertainty principle. That is

$$\Delta x \Delta p = \hbar \rightarrow r_{\min} = \frac{\hbar}{\Delta p} = \frac{\hbar}{\Delta mv} \approx \frac{\hbar}{m_e v_e}.$$

$\hbar$ is called Planck's constant and is = 1.054589E-34 J-s. The reason that we cannot use distance of closest approach like before is because for T ~ 10 keV, the distance of closest approach is about 144Z fm. Since Δp is about 100 keV/c, then Δx is about 2000 fm which is much larger than the distance of closest approach. So we cannot specify where the electron actually is. So for plasma with ion density of n_i the power loss per unit volume of plasma due to Bremsstrahlung radiation is

$$P_{br} = \int_{r_{\min}}^{r_{\max}} \frac{4\pi e^6 Z^2 n_e n_i}{3(4\pi\varepsilon_0)^3 c^3 m_e^2 r^2} dr = \frac{4\pi e^6 Z^2 n_i n_e v_e}{3(4\pi\varepsilon_0)^3 c^3 m_e \hbar}.$$

Numerically, this is

$$P_{br} = 4.522 \times 10^{-37} Z^2 n_i n_e (k_B T_e)^{1/2} \text{ Pa/s or W/m}^3,$$

where $v_e \approx \sqrt{3k_B T_e / m_e}$ and k_BTe is in keV. If there are more than one species of ions, then

$$\boxed{P_{br} = 4.522 \times 10^{-37} n_e (k_B T_e)^{1/2} \sum_j Z_j^2 n_j}$$

Remarks

- The larger Z, the higher P_{br}. So we want to keep Z as low as possible. Also, impurities can affect overall performance. For example, assume we have 1% of Oxygen impurity,

$$\begin{aligned}\frac{P_{br}(with\ oxygen)}{P_{br}(without\ oxygen)} &= \frac{T_e^{1/2} n_D^2 (1+\xi Z_{Ox})(1+\xi Z_{Ox}^2)}{T_e^{1/2} n_D^2} \\ &= 1 + \xi Z_{Ox} + \xi Z_{Ox}^2 + \xi^2 Z_{Ox}^3 \\ &= 1 + (0.01)(8) + (0.01)(8^2) + (0.01^2)(8^3) \\ &= 1.77\end{aligned}$$

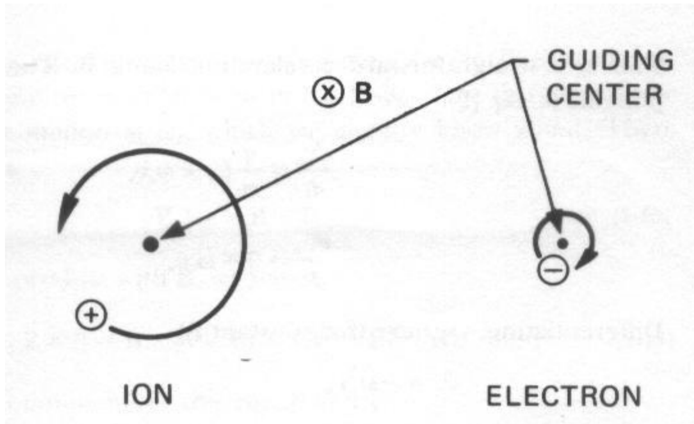
  Bremsstrahlung loss increases by 77% due to only 1% of Oxygen impurity!

- See figure 14.7 Krane. We can see that D-T reaction requires much less temperature than D-D reaction to overcome loss by Bremsstrahlung radiation (4 keV vs. 40 keV).

- Bremsstrahlung radiation is a soft x-ray. It cannot be absorbed by plasma, so when it hits the wall, it's considered loss. We can take advantage of this to measure T_e, n_e, and impurity in plasma by measure soft x-ray at the wall.

Another type of loss that is less severed is through cyclotron radiation that is due to electron moving in magnetic field.

An electron in a magnetic field moves in circular direction with frequency (draw electron rotation)



Larmor orbits in a magnetic field.

$$\omega_c = \frac{eB}{m_e}$$

Since in circular motion, $v = \omega r$ and $\dot{v} = \omega_c^2 r = \omega_c v$, then once again power radiated by electron being accelerated is

$$P = \frac{e^2 |\dot{v}|^2}{6\pi\varepsilon_0 c^3} = \frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{eB}{m_e} v\right)^2 = \frac{e^4 B^2 v^2}{6\pi\varepsilon_0 c^3 m_e^2} .$$

If we take v = thermal v, then for 2-D velocity distribution

$$v_{th} = \sqrt{\frac{2k_B T_e}{m_e}} .$$

The reason that we only take 2-D is because electrons rotate the same way in constant B field. Then we can write for electron density n_e

$$P_{cyc} = \frac{e^4 B^2 k_B T_e n_e}{3\pi\varepsilon_0 c^3 m_e^3} .$$

So what is B in fusion plasma?

Pressure due to magnetic field in plasma is (to be derived later when we get into fluid part)

$$P_{magnetic} = \frac{B^2}{2\mu_0}$$

Kinetic pressure of plasma

$$P_{kinetic} = n_e k_B T_e + n_i k_B T_i$$

Define $\beta = \frac{Kinetic\ pressure}{Magnetic\ pressure}$. For plasma to be confined, magnetic pressure must be greater than kinetic pressure. So $\beta \le 1$. Let assume that $\beta = 1$ for marginally confined case. Also assume that n_e = n_i and T_e = T_i. We then have,

$$\frac{B^2}{2\mu_0} = 2n_e k_B T_e$$

$$B^2 = 4\mu_0 n_e k_B T_e.$$

Thus,

$$\boxed{P_{cyc} = \frac{4e^4 \mu_0 n_e^2 k_B^2 T_e^2}{3\pi\varepsilon_0 c^3 m_e^3} = \frac{4e^4 n_e^2 k_B^2 T_e^2}{3\pi\varepsilon_0^2 c^5 m_e^3}} \text{ for } c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Or $P_{cyc} = 4.966 \times 10^{-38} n_e^2 (k_B T_e)^2$ [Pa/s or W/m^3] where $k_B T_e$ is in keV and n_e is in m^{-3}.

Remarks

- As we can see, $P_{br} \propto n_e^2 T_e^{1/2}$ while $P_{cyc} \propto n_e^2 T_e^2$. So loss by cyclotron radiation is more sensitive to electron temperature than the Bremsstrahlung radiation.
- However, cyclotron radiation is normally absorbed back by plasma, and can be controlled by using magnetic field.
- When choosing operating power of fusion reaction, then we only need to offset the power loss by Bremsstrahlung radiation, e.g. 4 keV for D-T and 40 keV for D-D.

So how do we make fusion possible in laboratory?

We want the reaction rate to be as high as possible, so we need high density n, and high <σv>. We also need to maintain this condition for a long time too, so we can generate enough fusion power. Remember that the time scales for D-D and D-T fusion reactions are in the order of 4 hours and 140 sec respectively. The sun has huge gravitational force and high abundance of proton to keep reaction going, but we do not and cannot create such gravitational force on earth. So we need to do this artificially. There are 2 schemes currently being pursued: magnetic confinement and inertial confinement.