Last time

- Fusion reactions
- Exothermic reaction – release energy
  - Q value: $m(M) + m(N) - m(O) - m(P) = B(O) + B(P) - B(M) - B(N) \equiv Q$
- Light nuclei
  - Binding energy curve
  - Z dependence on Coulomb barrier: $V_C = \frac{e^2}{4\pi\varepsilon_0}\frac{Z_a Z_b}{R_a + R_b}$
  - Z dependence on Bremsstrahlung radiation loss
- Fusion reaction rate: $\langle \sigma v \rangle \propto \int_0^\infty \frac{1}{v} e^{-2G} e^{-\frac{1}{2}mv^2 / k_B T} v^2 dv = \int_0^\infty e^{-2G} e^{-E/k_B T} dE$
- Lawson's criterion: $n\tau > \frac{12 k_B T}{\langle \sigma v \rangle Q}$
- Competing processes
  - Bremsstrahlung radiation:
    - Most severe
    - $P_{br} \propto n_e^2 T_e^{1/2}$
    - Effect of impurity: $P_{br} = 4.522 \times 10^{-37} n_e (k_B T_e)^{1/2} \sum_j Z_j^2 n_j$
  - Cyclotron radiation: $P_{cyc} \propto n_e^2 T_e^2$
    - Can be effectively absorbed back into plasma
- Achieving fusion in laboratory
  - Need high temperature & high pressure
  - Current solution
    - Magnetic confinement: Tokamak

- Inertial confinement: NIF, Z-Pinch

Single particle motions in electric and/or magnetic fields

The most important equation which characterizes the movement of charged particle in electric and/or magnetic fields is called Lorentz force equation, or sometimes referred to as equation of motion:

$$\underline{F} = m\frac{d\underline{v}}{dt} = m\dot{\underline{v}} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

where the underscore indicates vector quantities. $\underline{E}$ and $\underline{B}$ are respectively the electric and magnetic fields, and can be uniform or varying in time. Since this is vector equation, directions of $\underline{E}$ and $\underline{B}$ can affect the resulting force acting in the particle. We will consider various cases.

Using Cartesian coordinate, we can write vector

$$\underline{W} = W_x\hat{x} + W_y\hat{y} + W_z\hat{z}$$

and

$$\underline{W} \times \underline{V} = \left[W_yV_z - W_zV_y\right]\hat{x} + \left[W_zV_x - W_xV_z\right]\hat{y} + \left[W_xV_y - W_yV_z\right]\hat{z}$$

A. Uniform E and B fields (d/dx = d/dt = 0)

1. $\underline{E} = 0$, $\underline{B} \neq 0$

   Let $\underline{B} = B_z\hat{z}$, the Lorentz force equation is reduced to

$$m\dot{\underline{v}} = q\left(\underline{v} \times B_z\hat{z}\right).$$

   It is clear that v_z is a constant, because the cross product would be zero in z-direction. We shall call $v_z = v_{||}$ for which subscript || indicates that it is parallel to $\underline{B}$.

   Thus

$$m\left(\dot{v}_x\hat{x} + \dot{v}_y\hat{y}\right) = q\left(\left[v_x\hat{x} + v_y\hat{y}\right] \times B_z\hat{z}\right).$$

   So $$m\dot{v}_x\hat{x} = qv_yB_z\hat{x} \Rightarrow \dot{v}_x = \left(\frac{qB_z}{m}\right)v_y$$

$$m\dot{v}_y\hat{y} = -qv_xB_z\hat{y} \Rightarrow \dot{v}_y = -\left(\frac{qB_z}{m}\right)v_x$$

Differentiating both equations gives $$\ddot{v}_x = \left(\frac{qB_z}{m}\right)\dot{v}_y = -\left(\frac{qB_z}{m}\right)^2 v_x$$

$$\ddot{v}_y = -\left(\frac{qB_z}{m}\right)\dot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_y$$

Define **cyclotron frequency:** $$\boxed{\omega_c = \left|\frac{qB_z}{m}\right|}$$

Then we have the following set of equations

$$\dot{v}_x = \pm\omega_c v_y \qquad \ddot{v}_x = -\omega_c^2 v_x$$

$$\dot{v}_y = \mp\omega_c v_x \qquad \ddot{v}_y = -\omega_c^2 v_y$$

where $\pm$ indicates sign of qB_z (if qB_z is positive, we use the top sign; if qB_z is negative, we use the bottom sign). These are equations of oscillating system which have solutions

$$v_x = v_\perp \sin(\pm\,\omega_c t + \delta)$$

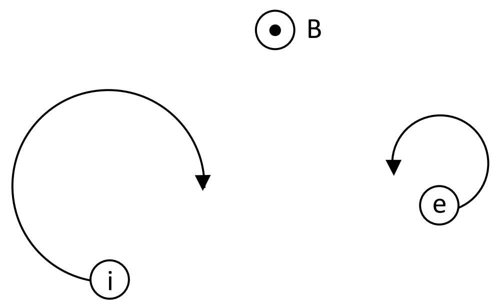
$$v_y = v_\perp \cos(\pm\,\omega_c t + \delta)$$

where $v_\perp$ is positive, and subscript $\perp$ indicates that it is perpendicular to $\underline{B}$. δ is a constant phase which we can conveniently choose to be zero. It follows that

$$\boxed{x - x_0 = \int v_x dt = \int v_\perp \sin(\pm\,\omega_c t + \delta)dt = \mp\frac{v_\perp}{\omega_c}\cos(\pm\,\omega_c t + \delta) = \mp r_L \cos(\pm\,\omega_c t + \delta)}$$

$$\boxed{y - y_0 = \int v_y dt = \int v_\perp \cos(\pm\,\omega_c t + \delta)dt = \pm\frac{v_\perp}{\omega_c}\sin(\pm\,\omega_c t + \delta) = \pm r_L \sin(\pm\,\omega_c t + \delta)}$$

where $\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \left|\frac{mv_\perp}{qB}\right|}$ is known as **Larmor radius**. If we plot the trajectory of the electron, it would look like this.



Remarks

1. These equations suggest that in a present of magnetic field, charge particle moves in circular orbit which centered at (x_0, y_0). This center is known as **guiding center**. The direction of the orbit depends on the sign of qB_z. So electron and ion behave differently.

2. We can use right-hand-rule to remember the rotation. With the thumb pointing to the direction of B, electron rotation follows the right-hand-rule, while ion rotation is against.

3. Since $r_L \propto m$, ion which has larger mass than electron would have larger rotation radius.

4. Also $r_L = \dfrac{v_\perp}{\omega_c} \propto \dfrac{T}{B}$.

5. Both electron and ion shown above produce current. The current then produces magnetic field B_s which is in the opposite direction from the applied B. Thus, plasma is **diamagnetic**.

2. $\underline{E} \neq 0$, $\underline{B} \neq 0$

Again, let $\underline{B} = B_z \hat{z}$. There are 2 cases worth considering here. One is when $\underline{E} \parallel \underline{B}$, and another is when $\underline{E} \perp \underline{B}$.

a. $\underline{E} \parallel \underline{B} \Rightarrow \underline{E} = E_z \hat{z}$

The Lorentz force equation is

$$m\dot{\underline{v}} = q\left(E_z \hat{z} + \underline{v} \times B_z \hat{z}\right).$$

We can see that the electric field E only affect the velocity component in the z-direction. So in the x- and y- directions, we can still write

$$v_x = v_\perp \sin(\pm \omega_c t + \delta)$$

$$v_y = v_\perp \cos(\pm \omega_c t + \delta)$$

and the same equations for x and y. However in the z-direction, we have

$$m\dot{v}_z = qE_z$$

$$\dot{v}_z = \frac{qE_z}{m}$$

$$v_z - v_{z0} = \frac{qE_z}{m} t$$

$$z - z_0 = \frac{qE_z}{2m} t^2 - v_{z0} t$$

which is a simple linear motion. Thus, while the charge particle rotates around the guiding center as in the first case, it has a linear motion in the z-direction superimposing on it.

Note that since electron and ion have opposite charge, the presence of this type of E field will cause **charge separation** which results in net current.

b. $\underline{E} \perp \underline{B} \Rightarrow \underline{E} = E_x \hat{x}$

Here we can write the 4 equations of motion as

$$\dot{v}_x = \pm \omega_c v_y + \frac{qE_x}{m} \qquad \ddot{v}_x = -\omega_c^2 v_x$$

$$\dot{v}_y = \mp \omega_c v_x \qquad \ddot{v}_y = -\omega_c^2 v_y \mp \omega_c \frac{qE_x}{m} = -\omega_c^2 v_y - \omega_c^2 \frac{E_x}{B_z}$$

where $\frac{qE_x}{m} = \frac{qB_z}{m}\frac{E_x}{B_z} = \pm \omega_c \frac{E_x}{B_z}$. So,

$$v_x = v_\perp \sin(\pm \omega_c t + \delta)$$

$$v_y = v_\perp \cos(\pm \omega_c t + \delta) - \frac{E_x}{B_z}$$

Note that $-\frac{E_x}{B_z}\hat{y} = \frac{E_x\hat{x}\times B_z\hat{z}}{|B_z\hat{z}|^2} = \boxed{\frac{\underline{E}\times\underline{B}}{|\underline{B}|^2} \equiv \underline{v}_{ExB}}$. We call this term **ExB drift**.

Remarks

1. While charge particle rotates with cyclotron motion, it experiences ExB drift in the direction of $\underline{E}\times\underline{B}$. One can look at it as a cyclotron motion which has its guiding center moving at speed v_{ExB}.

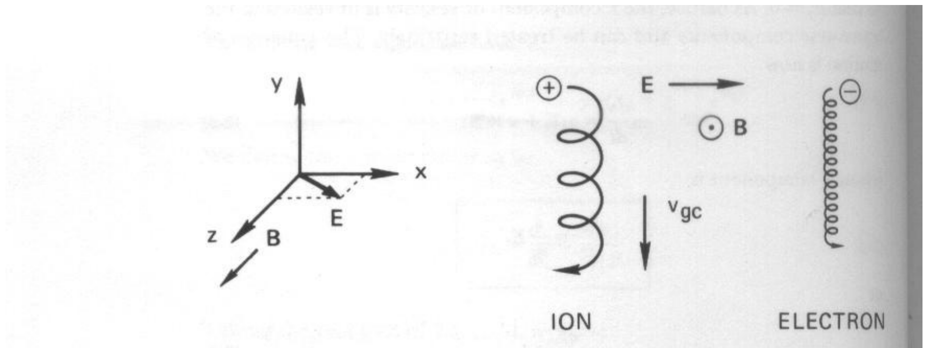
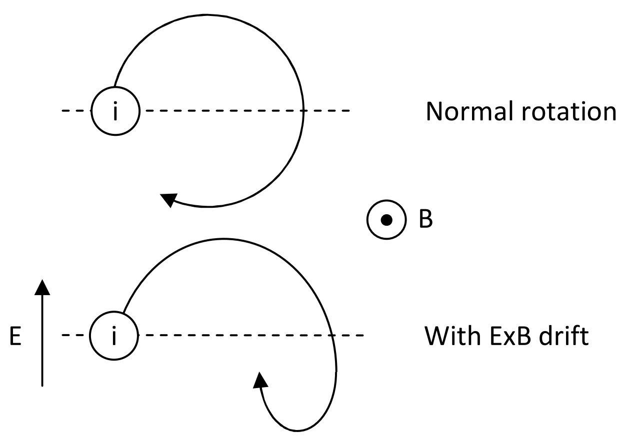


FIGURE 2-2 **Particle drifts in crossed electric and magnetic fields.**

2. While the charge particle rotates, half of it experiences gain in energy from E field, so r_L gets larger. The other half, however, experiences lose of energy, so r_L gets smaller. See example of ion below. Electron gains and loses energy in the opposite half, but because it also rotates in the opposite direction, the resulting drift is the same.

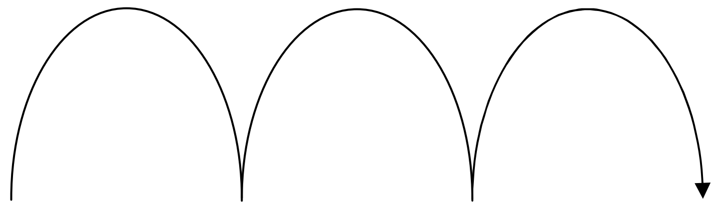


3. The ExB drift does not depend on charge of the particle. Electron and ion moves in the same direction with the same velocity causing **NO charge separation** = no net current.

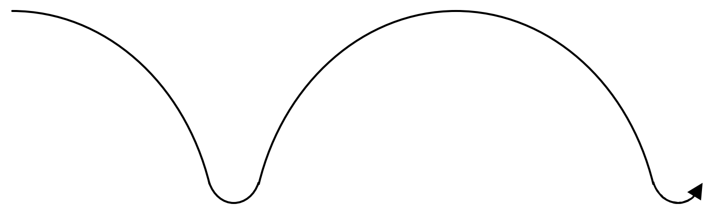
4. Notice that if

a. $v_\perp > v_{ExB}$, we still get the rotation as shown before.

b. $v_\perp = v_{ExB}$, we get half rotation instead.



c. $v_\perp < v_{ExB}$, we no longer get a full rotation.



3. A general case where we have external uniform force F acting on charge particle in the presence of uniform magnetic field B

   Again, there are 2 cases

   a. $\underline{F} \parallel \underline{B}$

      Here it will be like case 2a except that there will be **NO charge separation** because F does not depend on charge.

   b. $\underline{F} \perp \underline{B}$

      Since $q\underline{E} = \underline{F}$, we can replace $\underline{E}$ in 2b by $\underline{F}/q$. Thus, the charge experiences FxB drift where

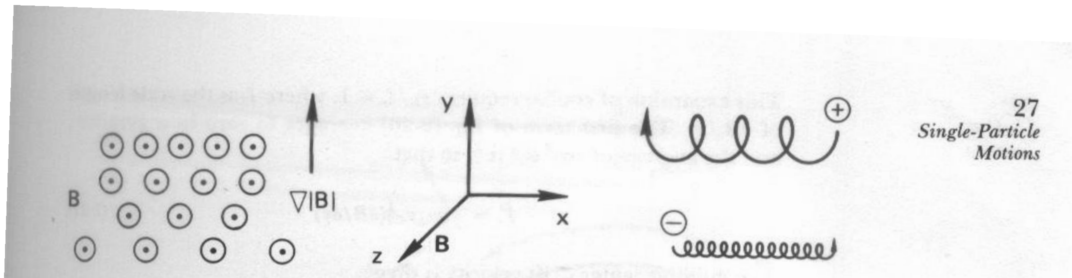
$$\boxed{\underline{v}_{F\times B} = \frac{\underline{F}\times\underline{B}}{q|\underline{B}|^2}}$$

      Since v_{FxB} now depends on charge, there will be **charge separation**, hence net current, between ion and electron.

B. Spatially varying E or B fields (d/dt = 0, but d/dx is not)

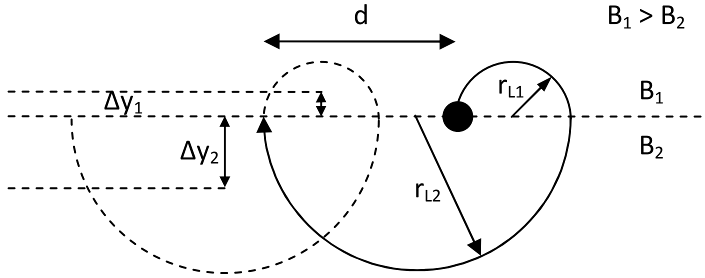
1. $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{\nabla} B \neq 0$

Let $\underline{B} = B_z(y)\hat{z}$ and $\underline{\nabla} B = \dfrac{dB_z(y)}{dy}\hat{y}$. Also, assuming that B_z increases in y-direction.



The drift of a gyrating particle in a nonuniform magnetic field. FIGURE 2-5

Qualitatively, as y gets larger, B_z gets stronger. Because r_L is inversely proportional to B, then if we separate the rotation of the charge particle to 2 half, top and bottom, r_L in the top half will be smaller than r_L in the bottom half.



From the figure, we have

$$d = 2r_{L2} - 2r_{L1}$$

Group velocity v_g is

$$v_g = \frac{\text{distance}}{\text{time}} = \frac{d}{\frac{1}{2}\left(\frac{1}{f_1} + \frac{1}{f_2}\right)} = \frac{2r_{L2} - 2r_{L1}}{\frac{1}{2}\left(\frac{2\pi}{\omega_{c1}} + \frac{2\pi}{\omega_{c2}}\right)}$$

$$= \frac{2r_{L2} - 2r_{L1}}{\frac{2\pi}{\omega_c}} \quad ; \quad \frac{2\pi}{\omega_c} = \frac{1}{2}\left(\frac{2\pi}{\omega_{c1}} + \frac{2\pi}{\omega_{c2}}\right)$$

$$= \frac{\omega_c}{\pi}\left[\frac{v_\perp}{\omega_{c2}} - \frac{v_\perp}{\omega_{c1}}\right] = \frac{\omega_c v_\perp}{\pi}\left[\frac{1}{\omega_{c2}} - \frac{1}{\omega_{c1}}\right]$$

$$= \frac{\omega_c v_\perp}{\pi}\left[\frac{\omega_{c1} - \omega_{c2}}{\omega_{c2}\omega_{c1}}\right]$$

$$\boxed{\begin{aligned} v_g &= \frac{v_\perp}{\pi}\left[\frac{\Delta\omega_c}{\omega_c}\right] \quad ; \quad \omega_c \approx \omega_{c1} \approx \omega_{c2} \\ &= \frac{v_\perp}{\pi}\left[\frac{\Delta B}{B}\right] \end{aligned}}$$

We can write

$$\Delta B = \frac{dB}{dy}\Delta y = \left|\underline{\nabla} B\right|\Delta y$$

where $\Delta y = \Delta y_1 + \Delta y_2$

$$\Delta y_1 = \frac{\int_{-r_{L1}}^{+r_{L1}} y\,dx}{\int_{-r_{L1}}^{+r_{L1}} dx} = \frac{\int_{-r_{L1}}^{+r_{L1}} \sqrt{r_{L1}^2 - x^2}\,dx}{\int_{-r_{L1}}^{+r_{L1}} dx} = \frac{\pi}{4} r_{L1}$$

and similarly, $\Delta y_2 = \frac{\pi}{4} r_{L2}$.

So, $\Delta y = \frac{\pi}{4}\left(r_{L1} + r_{L2}\right) = \frac{\pi}{2} r_L$

and $\Delta B = \frac{\pi}{2} r_L \left|\underline{\nabla} B\right|$

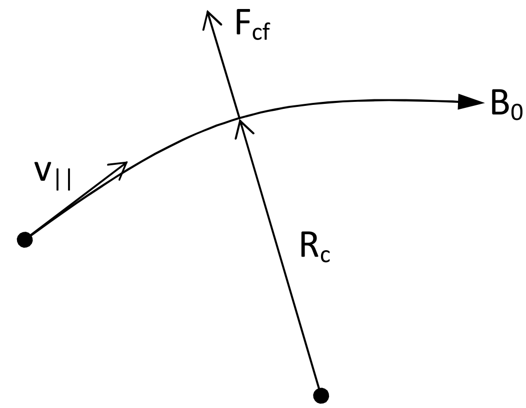
consequently,
$$\begin{aligned} v_g &= \frac{v_\perp}{\pi}\frac{\pi}{2} r_L \left[\frac{\left|\underline{\nabla} B\right|}{B}\right] \\ &= \frac{v_\perp^2}{2\omega_c}\left[\frac{\left|\underline{\nabla} B\right|}{B}\right] \end{aligned}$$

$$\boxed{\underline{v}_{\nabla B} = \pm\frac{v_\perp^2}{2\omega_c}\left[\frac{\underline{B}\times\underline{\nabla} B}{\left|\underline{B}\right|^2}\right]}$$

This drift is called **Grad-B drift**.

The $\pm$ sign comes from the fact that electron and ion rotates differently, thus they will move in opposite direction. We have left out this point since we started writing equation for v_g. We could have write $\pm$ there because the distance d for electron and ion have opposite sign. It is easily seen by redrawing the figure above for different charges.

2. $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{B}$ is curved



We can see that

$$\left|\underline{F}_{cf}\right| = \frac{mv_{\parallel}^2}{R_c}$$

and from before when we have $\underline{F} \perp \underline{B}$,

$$\underline{v}_{F\times B} = \frac{\underline{F}\times\underline{B}}{q\left|\underline{B}\right|^2}.$$

F_{cf} is in r-direction while B is in θ-direction

$$\underline{v}_{curve} = \underline{v}_{F\times B} = \frac{\left(\dfrac{mv_{\parallel}^2}{R_c}\right)\hat{r}\times B_0\hat{\theta}}{q\left|B_0\hat{\theta}\right|^2}$$

$$= \frac{mv_{\parallel}^2}{q\left|\underline{B}\right|^2}\frac{\underline{R}_c\times\underline{B}}{R_c^2}$$

$$= \pm\frac{v_{\parallel}^2}{\omega_c\left|\underline{B}\right|}\frac{\underline{R}_c\times\underline{B}}{R_c^2}$$

We take notice that there is $\underline{\nabla} B$ in the negative r-direction.

Maxwell's equation (to be discussed later) also tells us that $\underline{\nabla}\times\underline{B} = 0$.

Thus, $\underline{\nabla}\times\underline{B}$ only has z-component, and $\underline{\nabla}\times\underline{B} = \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = 0$.

So, $B_\theta \propto 1/r$. Then

$$\frac{\nabla B}{|\underline{B}|} = -\frac{\underline{R}_c}{R_c^2}.$$

Consequently,

$$\underline{v}_{curve} = \pm \frac{v_\parallel^2}{\omega_c |\underline{B}|} \frac{\underline{R}_c \times \underline{B}}{R_c^2} = \mp \frac{v_\parallel^2}{\omega_c} \frac{\underline{\nabla B} \times \underline{B}}{|\underline{B}|^2} = \pm \frac{v_\parallel^2}{\omega_c} \frac{\underline{B} \times \underline{\nabla B}}{|\underline{B}|^2}$$

and from the grad-B drift,

$$\underline{v}_{\nabla B} = \pm \frac{v_\perp^2}{2\omega_c} \left[\frac{\underline{B} \times \underline{\nabla B}}{|\underline{B}|^2} \right] = \pm \frac{v_\perp^2}{2\omega_c |\underline{B}|} \frac{\underline{R}_c \times \underline{B}}{R_c^2}$$

Thus, total drift in curved B field is

$$\boxed{\underline{v}_{curve} + \underline{v}_{\nabla B} = \pm \frac{\underline{B} \times \underline{\nabla B}}{\omega_c |\underline{B}|^2} \left[v_\parallel^2 + \frac{v_\perp^2}{2} \right] = \pm \frac{\underline{R}_c \times \underline{B}}{\omega_c |\underline{B}| R_c^2} \left[v_\parallel^2 + \frac{v_\perp^2}{2} \right]}$$