Last time

Single particle motion in electric/magnetic fields

- Lorentz force equation: $\underline{F} = m\frac{d\underline{v}}{dt} = m\underline{\dot{v}} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$
- Uniform E and B
  - $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{B} = B_z\hat{z}$
    - Cyclotron frequency: $\omega_c = \left|\frac{qB_z}{m}\right|$
    - Larmor radius: $r_L \equiv \frac{v_\perp}{\omega_c} = \left|\frac{mv_\perp}{qB}\right|$
    - Charge particle rotating in circle perpendicular to B: RHR for electron, LHR for ion
    - Guiding center
    - Diamagnetism: self-generated B cancels imposed B
  - $\underline{E} \neq 0$, $\underline{B} \neq 0$, $\underline{B} = B_z\hat{z}$
    - $\underline{E} \parallel \underline{B} \Rightarrow \underline{E} = E_z\hat{z}$
      - Guiding center moves in Z direction
      - Electron and ion move in opposite direction resulting in charge separation
    - $\underline{E} \perp \underline{B} \Rightarrow \underline{E} = E_x\hat{x}$
      - ExB drift: $\underline{v}_{ExB} \equiv \frac{\underline{E} \times \underline{B}}{\left|\underline{B}\right|^2}$
      - Electron and ion move in the same direction resulting in no charge separation
  - $\underline{F} \neq 0$, $\underline{B} \neq 0$, $\underline{B} = B_z\hat{z}$
    - $\underline{F} = q\underline{E}$
    - $\underline{F} \parallel \underline{B} \Rightarrow \underline{F} = F_z\hat{z}$
      - Guiding center moves in Z direction

        - Electron and ion move in the same direction resulting in no charge separation
    - $\underline{F} \perp \underline{B} \Rightarrow \underline{F} = F_x \hat{x}$
        - ExB drift: $\underline{v}_{FxB} \equiv \dfrac{\underline{F} \times \underline{B}}{q|\underline{B}|^2}$
        - Electron and ion move in the opposite direction resulting in charge separation

- Spatially varying E or B fields
    - $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{B} = B_z(y)\hat{z} \Rightarrow \underline{\nabla}B \perp \underline{B}$
        - Grad-B drift: $\underline{v}_{\nabla B} = \pm \dfrac{v_\perp^2}{2\omega_c}\left[\dfrac{\underline{B} \times \underline{\nabla}B}{|\underline{B}|^2}\right]$
        - Electron and ion move in the opposite direction resulting in charge separation
    - $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{B} = B_\theta(r)\hat{\theta} \Rightarrow \underline{B}$ is curved, $\underline{\nabla}B \perp \underline{B}$
        - Consist of 2 components: curvature-B drift and grad-B drift
        - $\dfrac{\underline{\nabla}B}{|\underline{B}|} = -\dfrac{\underline{R}_c}{R_c^2}$
        - Curvature-B drift: $\underline{v}_{curve} = \pm \dfrac{v_\parallel^2}{\omega_c} \dfrac{\underline{B} \times \underline{\nabla}B}{|\underline{B}|^2} = \pm \dfrac{v_\parallel^2}{\omega_c|\underline{B}|} \dfrac{\underline{R}_c \times \underline{B}}{R_c^2}$
        - Grad-B drift: $\underline{v}_{\nabla B} = \pm \dfrac{v_\perp^2}{2\omega_c}\left[\dfrac{\underline{B} \times \underline{\nabla}B}{|\underline{B}|^2}\right] = \pm \dfrac{v_\perp^2}{2\omega_c|\underline{B}|} \dfrac{\underline{R}_c \times \underline{B}}{R_c^2}$
        - Total drift: $\underline{v}_{curve} + \underline{v}_{\nabla B} = \pm \dfrac{\underline{B} \times \underline{\nabla}B}{\omega_c|\underline{B}|^2}\left[v_\parallel^2 + \dfrac{v_\perp^2}{2}\right] = \pm \dfrac{\underline{R}_c \times \underline{B}}{\omega_c|\underline{B}|R_c^2}\left[v_\parallel^2 + \dfrac{v_\perp^2}{2}\right]$

Continue today on single particle motion

3. We have considered couple cases where $\underline{\nabla}B \perp \underline{B}$. Now if we have $\underline{\nabla}B \parallel \underline{B}$ instead so that $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{\nabla}B \parallel \underline{B}$ but small.

   Before we talk about this though, we need to understand couple definitions first.

- **Magnetic moment** of a charged particle orbit $\mu = IA = \left(\frac{e}{\tau}\right)\left(\pi r_L^2\right) = \left(\frac{e\omega_c}{2\pi}\right)\left(\frac{\pi v_\perp^2}{\omega_c^2}\right) = \frac{\frac{1}{2}mv_\perp^2}{B}$

  $= \frac{W_\perp}{B}$ where $W_\perp = \frac{1}{2}mv_\perp^2$.

- **Angular momentum** of charged particle in the presence of magnetic field B $= r_L m v_\perp = \frac{mv_\perp^2}{\omega_c}$

  $\propto \frac{W_\perp}{B} \propto \mu$. This term does not change much for a charged particle traveling along B as long as B doesn't change appreciately over a Larmor cycle, i.e. $r_L\left|\underline{\nabla} B\right| << B$ and $\frac{2\pi}{\omega_c}\frac{\partial B}{\partial t} << B$.

- **Magnetic flux** $= B\pi r_L^2 = B\pi \frac{v_\perp^2}{\omega_c^2} \propto \frac{v_\perp^2}{B} \propto \frac{W_\perp}{B} \propto \mu$. This term does not change much either if B does not change appreciately.

- That the charged particle rotates periodically but the total momentum $\oint pdq$ along q is constant even when B is changed slowly is called **adiabatic invariant**.

So how do these terms relate to what we are considering?

The total energy of the particle is,

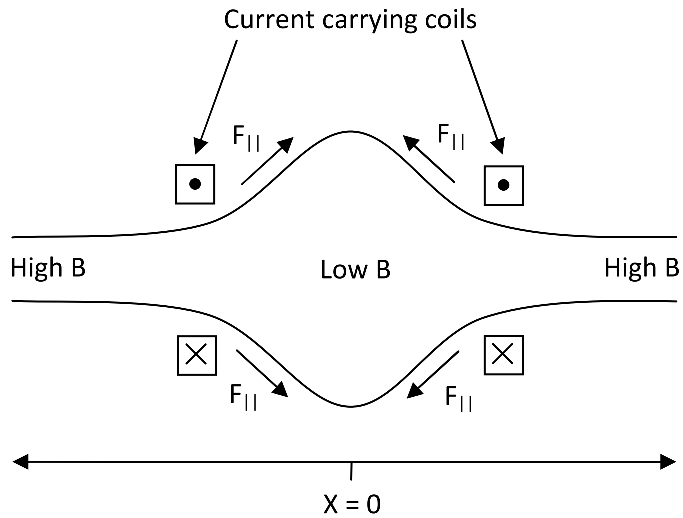
$$\begin{aligned} W &= W_\parallel + W_\perp \\ &= \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 \\ &= \frac{1}{2}mv_\parallel^2 + \mu B(r) \end{aligned}$$

where μB is called **mechanical potential**. Conservation of energy demands that $dW/dt = 0$ so that

$$\begin{aligned} mv_\parallel \frac{\partial v_\parallel}{\partial t} &= -\frac{\partial}{\partial t}\left[\mu B(r)\right] \\ m\frac{\partial v_\parallel}{\partial t} &= -\frac{\mu}{v_\parallel}\frac{\partial B(r)}{\partial t} = -\frac{\mu}{\partial s/\partial t}\frac{\partial B(r)}{\partial t} \\ m\frac{\partial v_\parallel}{\partial t} &= -\mu\frac{\partial B(r)}{\partial s} = F_\parallel \end{aligned}$$

where s = distance along magnetic field line. We can see that $F_{||}$ pointing against the direction of the change of B, i.e. $F_{||}$ points from high B region to low B region.

Consider **magnetic mirror**



The parallel forces keep particle confined between the current carrying coils. Even with this geometry, particles can still leak out through both ends. There are 3 cases that we should consider:

a. If $W_{\perp} = 0$, then $\mu = 0$ and $F_{\parallel} = 0$. Since μ is adiabatic invariant, then if it is found to be zero anywhere inside the mirror, it must be zero elsewhere including at both ends. Then particle cannot be trapped.

b. If $W_{\parallel} = 0$ at the center plane, then particle will be confined forever in the middle since it has no $v_{\parallel}$. This, however, usually not happen because of collision.

c. At the mirror's center, $v_{\perp}(center) = v_{\perp 0}$ and $v_{\parallel}(center) = v_{\parallel 0}$. If particle is trapped, then at the mirror's end $v_{\parallel}(end) = v_{\parallel 1} = 0$. $v_{\perp}$ at the mirror's end can be found from conservation of energy. That is, total energy at the mirror's center = total energy at the mirror's end. Then,

$$v_{\perp}(end) = v_{\perp 1} = \sqrt{v_{\parallel 0}^2 + v_{\perp 0}^2}\,.$$

Since μ is also conserved, and B(center) = B_{min} and B(end) = B_{max}, then

$$\frac{v_{\perp 0}^2}{B_{\min}} = \frac{v_{\parallel 0}^2 + v_{\perp 0}^2}{B_{\max}}.$$

We can define

- **mirror ratio R** = $B_{\max}/B_{\min}$, and
- **loss cone angle θ$_L$** = $\sin^{-1}\left(\sqrt{B_{\min}/B_{\max}}\right)$

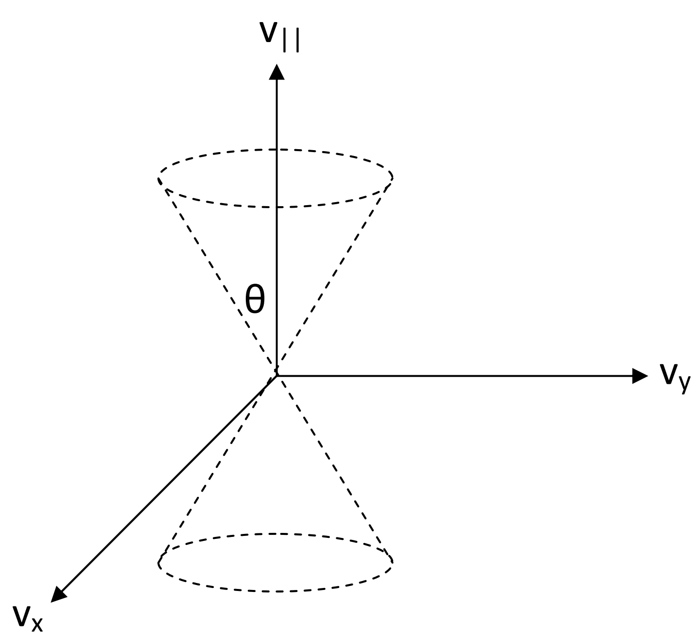
then

$$R = \frac{v_{\parallel 0}^2 + v_{\perp 0}^2}{v_{\perp 0}^2}$$

and

$$\theta_L = \sin^{-1}\left(\frac{1}{\sqrt{R}}\right) = \sin^{-1}\left(\sqrt{\frac{v_{\perp 0}^2}{v_{\parallel 0}^2 + v_{\perp 0}^2}}\right).$$

We can show this in velocity coordinate:



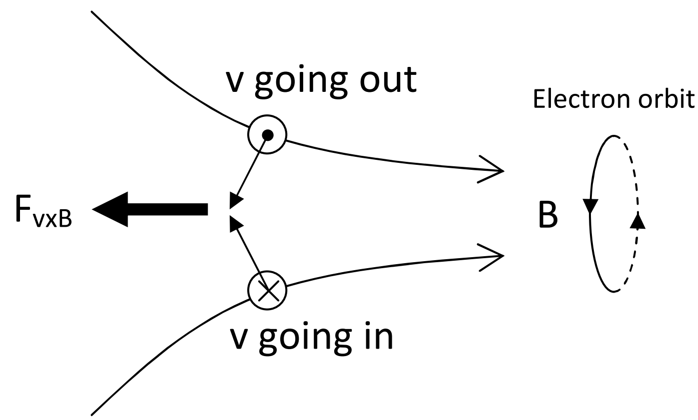
Let θ = pitch angle of the orbit = $\sin^{-1}\left(\sqrt{B_{\min}/B}\right) = \sin^{-1}\left(\sqrt{v_{\perp 0}^2/v_{\perp}^2}\right)$, then if $\theta < \theta_L$, particles are lost because B > B_{max} and the particles don't mirror back, and

if $\theta \geq \theta_L$, particles are confined because B <= B_{max} and the particles mirror back.

This is why θ_L is called "loss cone" angle, and the "cone" defines the boundary of the region outside which particles can be confined.

With collision, particles can scattered into loss cone and can no longer be confined. Electron which has higher collision frequency than ion is more likely to be lost.

d. A closer look at the magnetic mirror's end

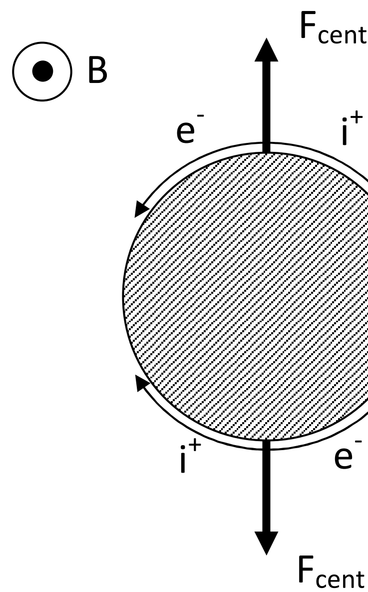


- Let's look at electron first. With B pointing to the right, the electron orbit follows the right-hand-rule.
- At the top, we have electron with velocity point out of plane, and vice versa at the bottom.
- The force at the top due to vxB is then pointing toward down-left (electron has negative charge).
- The force at the bottom due to vxB is pointing toward top-left.
- As a result, the total force points toward the left pushing electron back into the mirror
- For ion, the orbit is reversed. Velocity points inward at the top, and outward at the bottom.
- vxB forces at the top and bottom are the same as the forces acting on electron. Hence, the total force pushes ion back into the mirror as well.

e. Ideally, we should be able to confine particle using magnetic mirror. However, this doesn't work well in practice. Why?

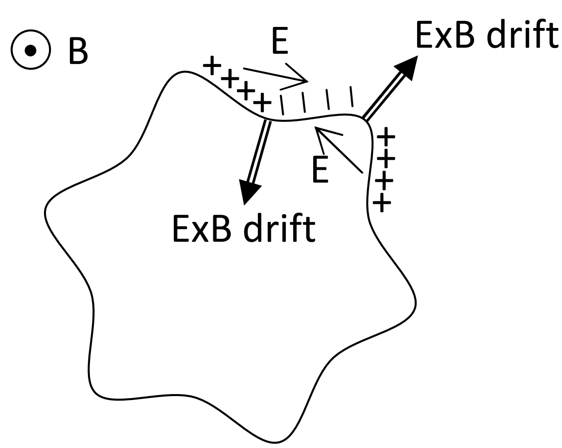
- For one reason, collisions cause some particle to escape as stated earlier.

- Second, this geometry is not in equilibrium, thus unstable.
  - Consider the middle cross section of the magnetic mirror, i.e. looking from the side.



Electrons and ions drift in opposite direction due to gradiant and curvature drifts.

  - If the system deviates from this, i.e. the system is perturbed. Then,



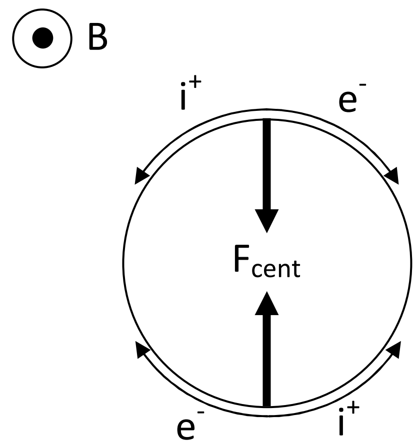
First, curvature drift separates ions and electrons.

Second, charge separation causes electric field to be set up.

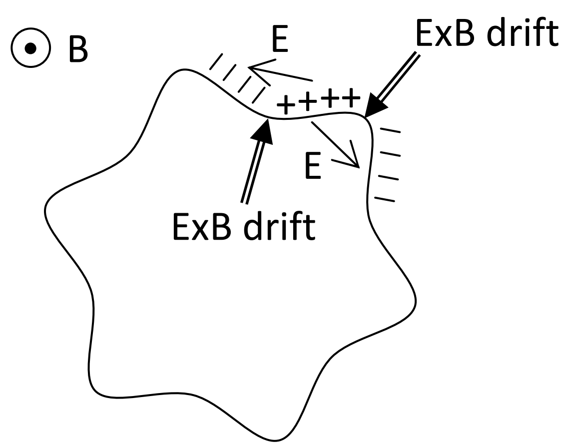
Third, ExB drift due to electric field reinforces the perturbation (inward/outward).

This kind of instability has many names such as **flute, interchange, gravitational, Rayleigh-Taylor instability** depending on the source of the instability.

- What about at the mirror's end? At the mirror's end, the direction of the centrifugal force is opposite, so the orbits of the electron and ion are opposite.

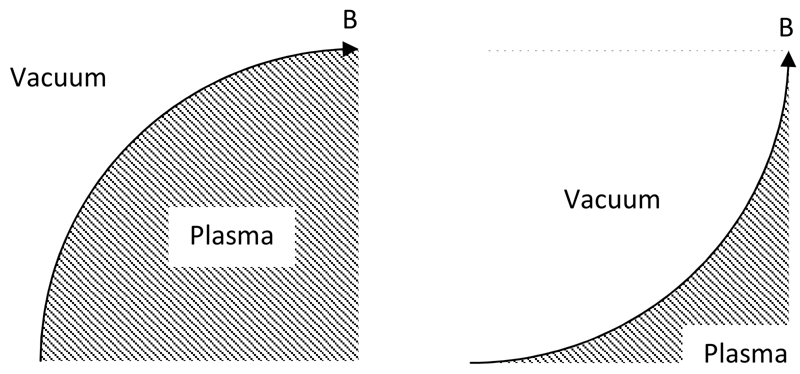


When there is perturbation, then



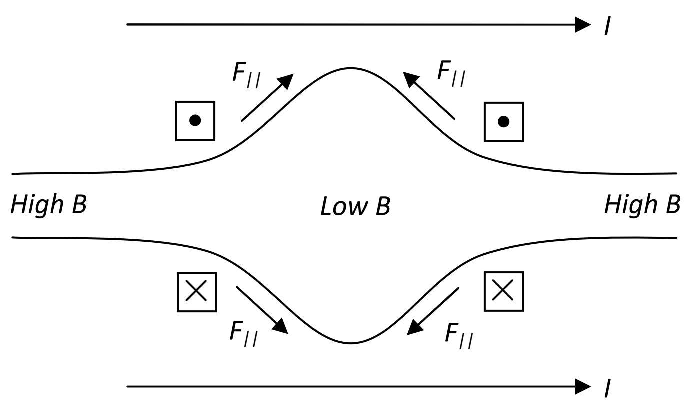
The resulting ExB drift acts against the initial perturbation, thus helps restoring the deviations.

- We can see then that at the mirror's center, the geometry is unstable. However, at the mirror's end, the geometry is stable.
- Notice that the stability depends on the curvature of the magnetic field around plasma that we try to confine.



Unstable Stable

- To help with confinement at the middle of the mirror, Ioffe bars, which is essentially current lines, are used to create strong B field around to push plasma back in, hence, minimum-B geometry.



4. Non-uniform E field.

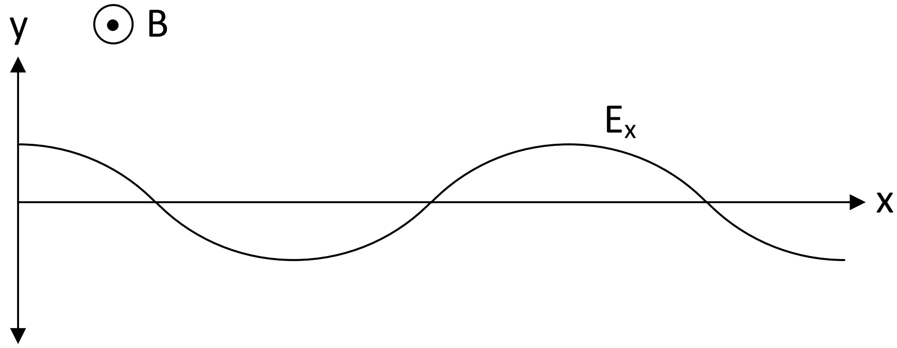
   Now let's consider charged particle's motion in a spatially varying E field while B field is uniform.

   Let's

$$\underline{E} = E_0 (\cos kx)\hat{x}.$$

   That is, both E and its variation is in the x direction. Also, let

$$\underline{B} = B_0 \hat{z}$$



We can see that E changes direction, so the resulting ExB drift depends on the direction of E. For positive E, ExB drift would point in negative y-direction. For negative E, ExB drift would point in positive y-direction.

Write,

$$m\frac{\partial \underline{v}}{\partial t} = q\left(\underline{E}(x) + \underline{v} \times \underline{B}\right).$$

Then,

$$\dot{v}_x = \frac{qB}{m} v_y + \frac{q}{m} E_x(x) \qquad \dot{v}_y = -\frac{qB}{m} v_x$$

$$\ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}_x}{B}$$

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x}{B}$$

Let's assume that the fluctuation in E is small, i.e. small perturbation in electric field. We can then estimate the x position as the unperturbed orbit,

$$x = x_0 \mp r_L \cos(\pm \omega_c t).$$

Let's look at y-direction

$$\begin{aligned}\ddot{v}_y &= -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B_0} \cos kx \\ &= -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B_0} \cos k\left(x_0 \mp r_L \cos(\pm \omega_c t)\right)\end{aligned}$$

This is an oscillation term, thus its average over one cycle is zero, i.e. $\bar{\ddot{v}}_y = 0$. Then,

$$\bar{v}_y = -\frac{E_0}{B_0}\overline{\cos k\left(x_0 \mp r_L \cos(\pm \omega_c t)\right)}.$$

Let's look at the cosine term,

$$\begin{aligned}\cos k\left(x_0 \mp r_L \cos(\pm \omega_c t)\right) &= \cos(kx)\cos(\mp kr_L \cos(\pm \omega_c t)) - \sin(kx)\sin(\mp kr_L \cos(\pm \omega_c t)) \\ &= \cos(kx)\cos(kr_L \cos(\omega_c t)) \pm \sin(kx)\sin(kr_L \cos(\omega_c t))\end{aligned}$$

When $kr_L << 1$ for small Larmor radius, we can expand

$$\begin{aligned}\cos\varepsilon &= 1 - \frac{1}{2}\varepsilon^2 + \ldots \\ \sin\varepsilon &= \varepsilon + \ldots\end{aligned}$$

Then,

$$\cos k\left(x_0 \mp r_L \cos(\pm \omega_c t)\right) = \cos(kx)\left(1 - \frac{1}{2}k^2 r_L^2 \cos^2(\omega_c t)\right) \pm \sin(kx) kr_L \cos(\omega_c t)$$

By averaging over time, the last term vanishes, we then get

$$\bar{v}_y = -\frac{E_0}{B_0}\cos(kx)\left(1 - \frac{1}{4}k^2 r_L^2\right) = -\frac{E_x(x)}{B}\left(1 - \frac{1}{4}k^2 r_L^2\right).$$

Note that, $\overline{\cos^2(\omega_c t)} = \frac{1}{2}$.

We can then write

$$\bar{v}_y = v_E = \frac{\underline{E}\times\underline{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right).$$

In general, we can replace ik by ∇, which gives

$$\boxed{v_E = \left(1 + \frac{1}{4}r_L^2 \nabla^2\right)\frac{\underline{E}\times\underline{B}}{B^2}}.$$

- Since this is time-average, it's the guiding-center drift because the rotation is averaged out. Notice that this is a combination of ExB drift and a different drift, both in ExB direction.

- Ion has larger r_L than electron, thus eventually there will be charge separation -> **finite-Larmor-radius effect**.

- The self-generate E due to charge separation enhances the existing E causing it to grow indefinitely -> **microinstability**.

- This charge separating due to r_L is more serious than the case of grad-B drift because it is proportional to r_L^2 rather than just r_L. Also, with large k, hence short wavelength, the effect would be greater.

- This kind of spatially variation in E can happen during plasma-wave interaction.

C. Time-varying E (d/dt is not 0)

1. Slowly time-varying E, E is spatially uniform, B is both time and spatially uniform.

Suppose we have,

$$\underline{E} = E_0(t)\hat{x} \text{ and } \underline{B} = B_0\hat{z}.$$

Slowly time-varying = no significant change over one Larmor cycle, thus

$$\left(\frac{2\pi}{\omega_c}\right)\frac{dE}{dt} << E_0.$$

We have from before in the case of uniform B and E,

$$\ddot{v}_y + \omega_c^2 v_y = \mp\frac{q}{m}\omega_c E.$$

When E has time-component, there are 2 types of solution,

- Homogeneous solution:

$$(v_y)_h = v_\perp \cos(\pm\omega_c t + \delta).$$

- Particular solution:

  - To find particular solution, we can treat E as constant since it varies slowly,
  - Thus,

$$(v_y)_p = \mp\frac{q}{m}\frac{\omega_c}{\omega_c^2}E_0(t) = \mp\frac{q}{m}\frac{E_0(t)}{\omega_c}.$$

This is essentially ExB drift.

So, $v_y = v_\perp \cos(\pm\omega_c t + \delta) \mp \frac{q}{m}\frac{E_0(t)}{\omega_c}$. Substituting this back into equation for v_x, we have

$$
\begin{aligned}
\ddot{v}_x &= \pm\omega_c \dot{v}_y + \frac{q\dot{E}}{m} \\
&= -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) \mp \frac{q}{m}\frac{\dot{E}}{\omega_c} + \frac{q\dot{E}}{m} \\
&\approx -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) + \frac{q\dot{E}}{m} \\
&= -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) + \frac{q\dot{E}}{m} \\
&= -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) + \frac{qB}{m}\frac{\dot{E}}{B} \\
&= -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) \pm \omega_c \frac{\dot{E}}{B} \\
&= -\omega_c^2 v_\perp \sin(\pm\,\omega_c t + \delta) \pm \omega_c^2 \frac{\dot{E}}{\omega_c B}
\end{aligned}
$$

We can see that the first term is the rotation movement. The second term is named polarization drift which is the effect of the slowly time-varying E. That is,

$$\boxed{\underline{v}_p = \pm\frac{1}{\omega_c B}\frac{d\underline{E}}{dt}}$$

- This kind of drift is charge dependent, hence there will be charge separation.