Last time

- Magnetic moment $\mu = \frac{\frac{1}{2}mv_\perp^2}{B}$ is adiabatic invariant. That is, μ remains constant while B can change slowly.

- Mechanical potential = μB so $W = W_\parallel + W_\perp = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 = \frac{1}{2}mv_\parallel^2 + \mu B(r)$.

- $\underline{E} = 0$, $\underline{B} \neq 0$, $\underline{\nabla}B \parallel \underline{B}$ but small.
    - $F_\parallel = m\frac{\partial v_\parallel}{\partial t} = -\mu \frac{\partial B(r)}{\partial s}$, $F_\parallel$ points from high B to low B.
    - Magnetic mirror
        - $\frac{v_{\perp 0}^2}{B_{\min}} = \frac{v_{\parallel 0}^2 + v_{\perp 0}^2}{B_{\max}}$
        - Mirror ratio R = $B_{\max}/B_{\min}$.
        - Loss cone angle θ_L = $\sin^{-1}\left(\sqrt{B_{\min}/B_{\max}}\right)$
        - Stable vs. unstable geometries. Stable at the mirror's end, unstable at the mirror's center.

- Spatially varying E field: $\underline{E} = E_0(\cos kx)\hat{x}$, $\underline{B} = B_0\hat{z}$.
    - Finite Larmor radius effect causes drift $v_E = \left(1 + \frac{1}{4}r_L^2\nabla^2\right)\frac{\underline{E}\times\underline{B}}{B^2}$.
    - This drift is proportional to r_L^2, thus ion moves faster than electron resulting in charge separation and self electric field.
    - Microinstability – the self electric field reinforces external E field, and the drift and the total E field keep growing.

- Slowly time varying E field: $\underline{E} = E_0(t)\hat{x}$, $\underline{B} = B_0\hat{z}$, $\left(\frac{2\pi}{\omega_c}\right)\frac{dE}{dt} << E_0$
    - Polarization drift $\underline{v}_p = \pm\frac{1}{\omega_c B}\frac{d\underline{E}}{dt}$ which is charge dependent.

Plasmas as Fluids

- In the last couple lectures, we have discussed the affect of E and B fields on the motions of charge dparticles.
- We, however, did not discuss how the presence of charged particles affects E and B fields other than a short discussion on charge separation.
- In reality, E and B fields depend on the motions of charged particles.

$$\boxed{E, B \leftrightarrow \text{charged particle motions}}$$

- This becomes self-consistent problem that we have to solve.
- We also cannot follow every single particle in a plasma since there are usually not enough computer memory to track them all.
- It turns out that we can predict plasma's behavior using fluid theory.
- In fluid theory, individuality of particle is neglected.
- The differences between normal fluid theory and plasma fluid theory are that 1) plasma contains charged particles, and 2) collision is much less frequent in plasma.

1. E, B, and n interaction in plasma

A set of equations governs how E, B, and n interact with each other. This set of equations is called Maxwell's equations:

In a medium,

$$\underline{\nabla} \bullet \underline{D} = \sigma \qquad \text{(Gauss's law)}$$

$$\underline{\nabla} \bullet \underline{B} = 0 \qquad \text{(Gauss's law of magnetism)}$$

$$\underline{\nabla} \times \underline{E} = -\frac{d\underline{B}}{dt} \qquad \text{(Faraday's law)}$$

$$\underline{\nabla} \times \underline{H} = \underline{j} + \frac{d\underline{D}}{dt} \qquad \text{(Ampere's law)}$$

$$\underline{D} = \varepsilon \underline{E}$$

$$\underline{B} = \mu \underline{H}$$

Where E = electric field [V/m]

B = magnetic field [T]

D = electric flux density [C/m^2]

H = magnetic field intensity [A/m]

ε = permittivity [F/m]

μ = permeability [H/m]

σ = free charge density [C/m^3] = $\sum_i q_i n_i$

j = free current density [A/m^2] = $\sum_i q_i n_i v_i$

Recall that I have introduced Poisson's equation before, $\nabla^2\phi = -\frac{\sigma}{\varepsilon}$. This equation is easily derived from the Gauss's law by substituting $\underline{E} = -\underline{\nabla}\phi$. **This equality is only correct if $\underline{\nabla}\times\underline{E} = -\frac{d\underline{B}}{dt} = 0$, i.e. B is constant, since it follows from vector property that $\underline{\nabla}\times\underline{\nabla}\phi = 0$.**

2. Modified Lorentz force equation

Maxwell's equations are not enough to characterize plasma. We need to add effect of charged particle interaction with the fields too. This can be done through modified Lorentz's force equation. Recall that we had

$$m\frac{d\underline{v}}{dt} = q\left(\underline{E} + \underline{v}\times\underline{B}\right).$$

Assuming that a fluid of n particles moves at average speed $\underline{u}$, this becomes

$$mn\frac{d\underline{u}}{dt} = qn\left(\underline{E} + \underline{u}\times\underline{B}\right).$$

Here, $\frac{d\underline{u}}{dt}$ is the "total" time derivative of $\underline{u}$ in moving frame. We can rewrite it as

$$\frac{d\underline{u}}{dt} = \frac{\partial\underline{u}}{\partial t} + \left(\underline{u}\bullet\underline{\nabla}\right)\underline{u}$$

where $\frac{\partial \underline{u}}{\partial t}$ is the time derivative of $\underline{u}$ in fixed frame, and $(\underline{u} \bullet \underline{\nabla})\underline{u}$ is the convective term seen by observer moving with the fluid. The reason that we want to do this is because fluid doesn't hold its form all the time, so $\frac{d\underline{u}}{dt}$ is not easy to find. However, $\frac{\partial \underline{u}}{\partial t}$ which looks at the fluid change in time at a fixed point is a more practical quantity, i.e. imagine looking at a small volume of fluid element with fluid going in and out of it. This becomes

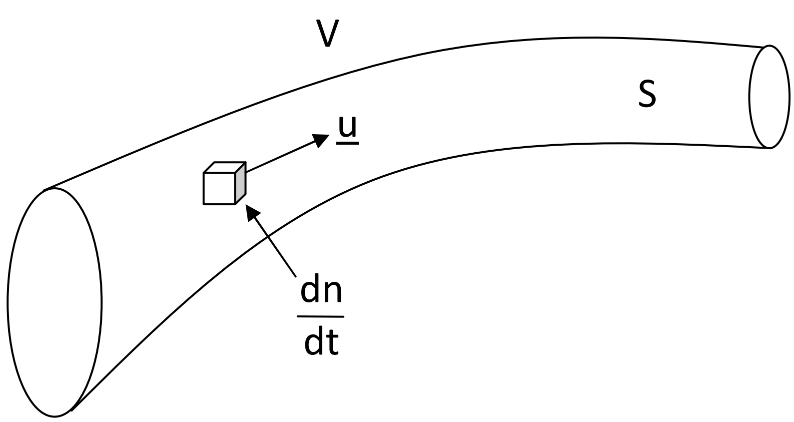
$$mn\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \bullet \underline{\nabla})\underline{u}\right] = qn(\underline{E} + \underline{u} \times \underline{B}).$$

3. Continuity equation

Assume that we have a section of fluid with volume V surrounded by surface S as shown below. Then,

rate of change in the total number of particles = $\int_V \frac{\partial n}{\partial t} dV$

net flux across boundary S = $-\oint_S n\underline{u} \bullet dS = -\int_V \underline{\nabla} \bullet (n\underline{u}) dV$



These 2 quantities are the same if there is no source or sink inside V to change number of particles. Thus, we have

$$\frac{\partial n}{\partial t} + \underline{\nabla} \bullet (n\underline{u}) = 0,$$

which is known as **continuity equation** or **equation of mass conservation**. It basically tells us that in the absence of source or sink, the change in the total number of particles inside a volume only happens through particle going in and out across the volume's boundary.

4. Pressure-gradient force

The modified Lorentz force equation

$$mn\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \bullet \underline{\nabla})\underline{u}\right] = qn(\underline{E} + \underline{u} \times \underline{B}),$$

however, is still not a complete form to describe plasma fluid. We need to include another type of force acting on the fluid element which comes from random motion of particles in the fluid.

- We have discussed before about plasma temperature, and velocity distribution.
- We know that when gas has velocity distribution is, there is pressure associated with it. That is,

$$p = nKT \; .$$

- The force associated with this pressure is

$$\underline{F}_p = -\underline{\nabla} p \; .$$

- This is the force that moves fluid or gas from high pressure to low pressure region. Wind is another example of the resulting of this force.
- Note that for cold fluid or low temperature plasma, $\nabla p = 0$ since there is no thermal motion or it is very small in comparison to other terms.

Thus, the complete form of plasma fluid equation is

$$mn\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \bullet \underline{\nabla})\underline{u}\right] + \underline{\nabla} p = qn(\underline{E} + \underline{u} \times \underline{B})$$

or

$$\boxed{mn\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \bullet \underline{\nabla})\underline{u}\right] = qn(\underline{E} + \underline{u} \times \underline{B}) - \underline{\nabla} p}$$

5. Equation of state

- Now, we have Maxwell's equations to relate E, B, and charge density; modified Lorentz force to relate E, B, v, and p; and continuity equation to relate n and v. To close this set of equations, we need to know what p is in term of the other parameters. That is equation of state.

- This equation of state is very a very important equation, and usually varies from system to system. Correctness of plasma calculations, both analytically and numerically, depend on using the right equation of state.

- There are many simple equations of state for basic plasma system. Three popular ones are

  a) Isotropic law: $p_{i,e} = n_{i,e} k_B T \Rightarrow \underline{\nabla} p = k_B T \underline{\nabla} n$

  b) Adiabatic law: $p_{i,e} = C n_{i,e}^{\gamma} \Rightarrow \underline{\nabla} p_{i,e} = C \gamma n_{i,e}^{\gamma-1} \underline{\nabla} n_{i,e} = \frac{\underline{\nabla} p_{i,e}}{p_{i,e}} = \gamma \frac{\underline{\nabla} n_{i,e}}{n_{i,e}}$ where C = constant, $\gamma = (2+N)/N$ = ratio of specific heat , and N = number of degree of freedom. Note that when $\gamma = 1$, we get isotropic law back.

  c) Fluid is incompressible: $\underline{\nabla} \bullet \underline{v}_{i,e} = 0$

6. Complete set

The complete set of fluid equations:

$$\underline{\nabla} \bullet \underline{D} = \sigma$$

$$\underline{\nabla} \bullet \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{d\underline{B}}{dt}$$

$$\underline{\nabla} \times \underline{H} = j + \frac{d\underline{D}}{dt}$$

$$\underline{D} = \varepsilon \underline{E}$$

$$\underline{B} = \mu \underline{H}$$

$$\frac{\partial n_i}{\partial t} + \underline{\nabla} \bullet (n_i \underline{v}_i) = 0$$

$$m_i n_i \left[\frac{\partial \underline{v}_i}{\partial t} + (\underline{v}_i \bullet \underline{\nabla}) \underline{v}_i \right] = q_i n_i (\underline{E} + \underline{v}_i \times \underline{B}) - \underline{\nabla} p_i$$

Equation of state

Subscript “i” is for each fluid species in plasma. For example, if there is one ion species, then we need to calculate 2 species, one for electron, and one for ion; if there are 2 ion species, then we need to calculate 3 species, one for electron, and two for the two ion species.

We, however, only need to separate ion and electron into 2 fluid species because they response differently, i.e. ion reaction time >> electron reaction time. However, when the time scale is long enough for ion to react with the fields, i.e. $\omega << \omega_{ci}$, we can combine ion and electron into one fluid. This kind of treatment is known as Magnetohydrodynamic or MHD treatment. The set of equations become:

$$\underline{\nabla} \bullet \underline{D} = \sigma$$

$$\underline{\nabla} \bullet \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{d\underline{B}}{dt}$$

$$\underline{\nabla} \times \underline{H} = \underline{j} + \frac{d\underline{D}}{dt}$$

$$\underline{D} = \varepsilon \underline{E}$$

$$\underline{B} = \mu \underline{H}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \bullet (\rho \underline{v}) = 0 \Rightarrow \text{ mass conservation law}$$

$$\frac{\partial \sigma}{\partial t} + \underline{\nabla} \bullet \underline{j} = 0 \Rightarrow \text{ charge conservation law}$$

$$\rho \frac{\partial \underline{v}}{\partial t} = \sigma \underline{E} + \underline{j} \times \underline{B} - \underline{\nabla} p$$

Equation of state

where $\rho = \sum_i n_i m_i$ and $\underline{v} = \frac{1}{\rho} \sum_i n_i m_i \underline{v}_i$.

This MHD equation is normally used for treating low frequency plasma.

7. Other effects

- The fluid equations which we have so far may still be incomplete. Other effects due to, for instance, gravitation and collisions, may come in and be large enough in comparison with other terms that we cannot neglect them.
- Effect of collisions
  - This can be due to the presence of neutral gas in the plasma. The amount of momentum loss depends on relative velocity between the fluid and the neutral gas.
  - In general, we can add another term, $-m_i n_i \nu_i (u_i - u_0)$, to the right-hand-side of the modified Lorentz force equation. Here ν_i is the ion-neutron collision frequency, and u_0 is velocity of the neutral fluid.
  - Charged particle collision can also be present. We will discuss about this later when we get to diffusion process in plasma.
- Effect of gravitation
  - We can add the grativational force term, $m_i n_i \underline{g}$, to the right-hand-side of the modified Lorentz force equation.

8. Klimontovich-Dupree Equation

- We have so far generalizes density of each fluid using parameter n. However, a better representation would be to consider the particle distribution in the fluid.
- Klimontovich-Dupree equation allows us to follow each particle in the fluid individually while preserves interaction between them, E, and B fields.
- First we can write density of particle in 6-dimensional phase space as

$$N(\underline{x},\underline{v},t) = \sum_i \delta(\underline{x} - \underline{X}_i(t))\delta(\underline{v} - \underline{V}_i(t))$$

  - This is a summation over all particles of the same species in the system or fluid.
  - 6 dimensions include 3 positional axes, and 3 velocity axes. It is actually 7 including time.
- Time differentiate the equation gives

$$\frac{\partial}{\partial t}N(\underline{x},\underline{v},t)=\sum_i\left[\delta(\underline{v}-\underline{V}_i(t))\frac{\partial}{\partial t}\delta(\underline{x}-\underline{X}_i(t))+\delta(\underline{x}-\underline{X}_i(t))\frac{\partial}{\partial t}\delta(\underline{v}-\underline{V}_i(t))\right]$$

$$=\sum_i\left[\begin{array}{l}\delta(\underline{v}-\underline{V}_i(t))\left(-\frac{\partial\underline{X}_i(t)}{\partial t}\bullet\underline{\nabla}_x\right)\delta(\underline{x}-\underline{X}_i(t))\\+\delta(\underline{x}-\underline{X}_i(t))\left(-\frac{\partial\underline{V}_i(t)}{\partial t}\bullet\underline{\nabla}_v\right)\delta(\underline{v}-\underline{V}_i(t))\end{array}\right]$$

$$=\sum_i\left[-\frac{\partial\underline{X}_i(t)}{\partial t}\bullet\underline{\nabla}_x-\frac{\partial\underline{V}_i(t)}{\partial t}\bullet\underline{\nabla}_v\right]\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

$$=\sum_i\left[-\underline{V}_i(t)\bullet\underline{\nabla}_x-\frac{\partial\underline{V}_i(t)}{\partial t}\bullet\underline{\nabla}_v\right]\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

where

$$\frac{\partial\underline{X}_i(t)}{\partial t}=\underline{V}_i(t)$$
$$\frac{\partial\underline{V}_i(t)}{\partial t}=\frac{q}{m}\left(\underline{E}^m(\underline{X}(t),t)+\underline{V}(\underline{X}(t),t)\times\underline{B}^m(\underline{X}(t),t)\right)$$

The subscript “m” indicate microscopic fields produced by the point particles themselves and the externally applied fields.

Since we have delta function here, the function only has non-zero value when $\underline{x}=\underline{X}_i(t)$ and $\underline{v}=\underline{V}_i(t)$. Thus, we can replace $\underline{X}_i(t)$ and $\underline{V}_i(t)$ by $\underline{x}$ and $\underline{v}$ respectively. This yields

$$\frac{\partial}{\partial t}N(\underline{x},\underline{v},t)=\sum_i\left[-\underline{v}\bullet\underline{\nabla}_x-\frac{q}{m}\left(\underline{E}^m(\underline{x},t)+\underline{v}\times\underline{B}^m(\underline{x},t)\right)\bullet\underline{\nabla}_v\right]\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

$$=\left[-\underline{v}\bullet\underline{\nabla}_x-\frac{q}{m}\left(\underline{E}^m(\underline{x},t)+\underline{v}\times\underline{B}^m(\underline{x},t)\right)\bullet\underline{\nabla}_v\right]\sum_i\delta(\underline{x}-\underline{X}_i(t))\delta(\underline{v}-\underline{V}_i(t))$$

$$=\left[-\underline{v}\bullet\underline{\nabla}_x-\frac{q}{m}\left(\underline{E}^m(\underline{x},t)+\underline{v}\times\underline{B}^m(\underline{x},t)\right)\bullet\underline{\nabla}_v\right]N(\underline{x},\underline{v},t)$$

$$\boxed{\frac{\partial}{\partial t}N(\underline{x},\underline{v},t)+\underline{v}\bullet\underline{\nabla}_xN(\underline{x},\underline{v},t)+\frac{q}{m}\left(\underline{E}^m(\underline{x},t)+\underline{v}\times\underline{B}^m(\underline{x},t)\right)\bullet\underline{\nabla}_vN(\underline{x},\underline{v},t)=0}$$

- This is the Klimontovich-Dupree equation. What it says is that density of particles of a single species is instant in time while moving along the path of an imaginary particle in phase space $\underline{x}$ and $\underline{v}$.

- The KD equation itself is still rather impractical since it is not in term particle distribution which we try to arrive at. In this form, it provides too much information more than we need.

- We can write the density N as

$$N(\underline{x},\underline{v},t) = f(\underline{x},\underline{v},t) + \delta N(\underline{x},\underline{v},t)$$

where $f(\underline{x},\underline{v},t)$ is the ensemble average of $N(\underline{x},\underline{v},t)$, and

$\delta N(\underline{x},\underline{v},t)$ is the remainder representing fluctuation which has **zero** ensemble average.

Similarly,

$$\underline{E}^{m}(\underline{x},t) = \underline{E}(\underline{x},t) + \delta \underline{E}(\underline{x},t)$$

$$\underline{B}^{m}(\underline{x},t) = \underline{B}(\underline{x},t) + \delta \underline{B}(\underline{x},t)$$

Substitute these into the KD equation and take ensemble average <>, we get

$$\begin{aligned}
&\frac{\partial}{\partial t} f(\underline{x},\underline{v},t) + \underline{v} \bullet \underline{\nabla}_x f(\underline{x},\underline{v},t) + \frac{q}{m}\left(\underline{E}(\underline{x},t) + \underline{v} \times \underline{B}(\underline{x},t)\right) \bullet \underline{\nabla}_v f(\underline{x},\underline{v},t) \\
&= -\frac{q}{m}\left\langle \left(\delta \underline{E}(\underline{x},t) + \underline{v} \times \delta \underline{B}(\underline{x},t)\right) \bullet \underline{\nabla}_v \delta N(\underline{x},\underline{v},t) \right\rangle \\
&\equiv \left.\frac{\partial f}{\partial t}\right|_{\text{collision}}
\end{aligned}$$

The right-hand-side of the equation is sensitive to discrete-particle nature of plasma which gives rise to collision, while the left-hand-side contains only particle distribution and no discrete-particle effect.

- If we treat plasma as continuum medium in phase space by breaking charge, says electron, into infinitesimal particle, and use "pulverization procedure" in which we assume that $n \to \infty$ but keeps en constant so that $e \to 0$. Also, $m \to 0$, but nm is finite. Then, in statistical mechanic

$$\delta n \sim \sqrt{n}$$

$$\delta E \sim e\delta n \propto e\sqrt{n} \propto \frac{1}{n}\sqrt{n} = \frac{1}{\sqrt{n}}$$

Then,

$$RHS \sim \delta E \cdot \delta n \sim \frac{1}{\sqrt{n}}\sqrt{n} \sim O(1)$$

$$LHS \sim n$$

We can see that LHS >> RHS, thus we can simply ignore RHS, i.e. collision effect, in continuum limit. Therefore, we arrive at

$$\frac{\partial}{\partial t} f(\underline{x}, \underline{v}, t) + \underline{v} \bullet \underline{\nabla}_x f(\underline{x}, \underline{v}, t) + \frac{q}{m}(\underline{E}(\underline{x}, t) + \underline{v} \times \underline{B}(\underline{x}, t)) \bullet \underline{\nabla}_v f(\underline{x}, \underline{v}, t) = 0$$

which is the well-known Vlasov equation, a.k.a. collisionless Boltzmann equation.

9. Vlasov equation

Vlasov equation is probably the most important equation in plasma physics. Many plasma property and phenomena can be derived from it.

Jean's Theorem states that $f = f(\alpha_1, \alpha_2, ..., \alpha_n)$ is a solution of Vlasov equation if α_1, α_2, ..., α_n are constants of motion on trajectories in phase space.

$$\frac{Df}{Dt} = \frac{D\alpha_1}{Dt}\frac{\partial f}{\partial \alpha_1} + \frac{D\alpha_2}{Dt}\frac{\partial f}{\partial \alpha_2} + ... + \frac{D\alpha_n}{Dt}\frac{\partial f}{\partial \alpha_n}$$

If α_1, α_2, ..., α_n are constants of motion, then

$$\frac{D\alpha_1}{Dt} = \frac{D\alpha_2}{Dt} = ... = 0$$

Consequently,

$$\frac{Df}{Dt} = 0$$

Examples,

a) When $\underline{E} = 0$ and $\underline{B} = 0$, then $\underline{v}$ is a constant of motion. So, $f(\underline{x}, \underline{v}, t) = f(\underline{v})$.

b) When $\underline{E} = 0$ and $\underline{B} = B_0 \hat{z}$ = constant. $v_{\parallel}$, $v_{\perp}$, and v_{gc} are constant of motion. Thus, $f = f(v_{\parallel}, v_{\perp}, v_y + \omega_c x)$.

c) $\underline{B} = 0$ and $\underline{E} = -\underline{\nabla}\phi(\underline{x})$. $\frac{1}{2}mv_x^2 + e\phi$ is constant of motion. Then, $f = f\left(\sqrt{v_x^2 + \frac{2}{m}e\phi}, v_y, v_z\right)$.