Last Time

- Plasma fluid discussion
- Construction of plasma fluid equations
- E, B, and n interaction in plasma
    - Maxwell's equations
- Modified Lorentz force equation
- Continuity equation
- Pressure-gradient force
- Equation of state
- Complete set of fluid equations

$$\underline{\nabla} \bullet \underline{D} = \sigma$$

$$\underline{\nabla} \bullet \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{d\underline{B}}{dt}$$

$$\underline{\nabla} \times \underline{H} = j + \frac{d\underline{D}}{dt}$$

$$\underline{D} = \varepsilon \underline{E}$$

$$\underline{B} = \mu \underline{H}$$

$$\frac{\partial n_i}{\partial t} + \underline{\nabla} \bullet (n_i \underline{v}_i) = 0$$

$$m_i n_i \left[\frac{\partial \underline{v}_i}{\partial t} + (\underline{v}_i \bullet \underline{\nabla}) \underline{v}_i \right] = q_i n_i (\underline{E} + \underline{v}_i \times \underline{B}) - \underline{\nabla} p_i$$

Equation of state

- Multiple fluid equations vs. Single fluid (Magnetohydrodynamic) equation
- Other effects on fluid equations

- Klimomtovich-Dupree equation

$$\frac{\partial}{\partial t}N(\underline{x},\underline{v},t)+\underline{v}\bullet\underline{\nabla}_x N(\underline{x},\underline{v},t)+\frac{q}{m}\left(\underline{E}^m(\underline{x},t)+\underline{v}\times\underline{B}^m(\underline{x},t)\right)\bullet\underline{\nabla}_v N(\underline{x},\underline{v},t)=0$$

- Pulverization procedure for continuum medium

$n\to\infty$ but keeps en constant so that $e\to 0$. Also, $m\to 0$, but nm is finite

- Vlasov equation, a.k.a. collisionless Boltzmann equation

$$\frac{\partial}{\partial t}f(\underline{x},\underline{v},t)+\underline{v}\bullet\underline{\nabla}_x f(\underline{x},\underline{v},t)+\frac{q}{m}\left(\underline{E}(\underline{x},t)+\underline{v}\times\underline{B}(\underline{x},t)\right)\bullet\underline{\nabla}_v f(\underline{x},\underline{v},t)=0$$

- Jean's theorem, and constant of motion

Jean's Theorem states that $f=f(\alpha_1,\alpha_2,...,\alpha_n)$ is a solution of Vlasov equation if α_1, α_2, …, α_n are constants of motion on trajectories in phase space.

Constant of Motion (Continue)

Let's look at the case when we have constant electric field and no B field. Let $\underline{E}=E_0\hat{x}$.

Then, the trajectory will be constant of motion in y and z directions because there is no acceleration due to electric field in that direction.

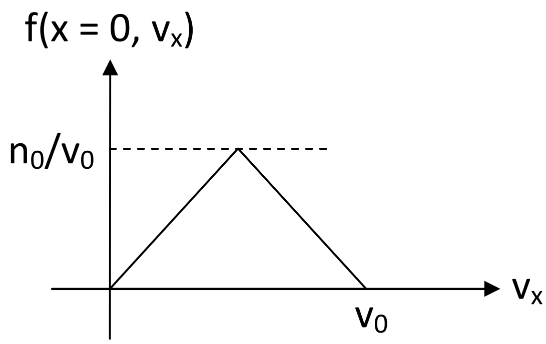
However in x direction, we can write conservation of energy as

$$\frac{1}{2}mv_x^2-eE_0x=\text{constant}$$

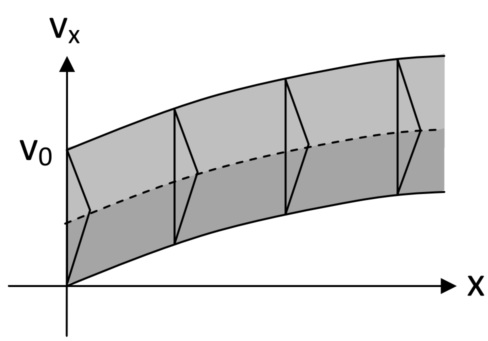
$$\sqrt{v_x^2-eE_0x}=\text{constant}\ .$$

Jean's theory says that $f\left(\sqrt{v_x^2-eE_0x},v_y,v_z\right)$ is a solution to Vlasov equation.

If we have the following profile in x direction,



Then if we plot the trajectory on x-v_x coordinate, we will have



Because of the constant of motion, the profile along the trajectory will remain the same.

Moments of Boltzmann Equation

Last time, we have

$$\underbrace{\frac{\partial}{\partial t} f(\underline{x},\underline{v},t)}_{\textbf{Term 1}} + \underbrace{\underline{v} \bullet \underline{\nabla}_x f(\underline{x},\underline{v},t)}_{\textbf{Term 2}} + \underbrace{\frac{q}{m}\left(\underline{E}(\underline{x},t) + \underline{v} \times \underline{B}(\underline{x},t)\right) \bullet \underline{\nabla}_v f(\underline{x},\underline{v},t)}_{\textbf{Term 3}} = \underbrace{\begin{cases} \left.\dfrac{\partial f}{\partial t}\right|_{\text{collision}} \\ 0 \end{cases}}_{\textbf{Term 4}}$$

The top term is called Boltzmann equation. The bottom term is called Vlasov equation or collisionless Boltzmann equation. There are 4 terms in it.

Also, we can write density *n* as

$$n(\underline{x},t) = \int_{-\infty}^{+\infty} f(\underline{x},\underline{v},t)\, d\underline{v}$$

where *n* is a function of position and time. Then, any average over this velocity distribution function $\underline{\Psi}(\underline{x}, \underline{v}, t)$ can be calculated by

$$\langle \underline{\Psi}(\underline{x}, t) \rangle = \frac{1}{n(\underline{x}, t)} \int_{-\infty}^{+\infty} \underline{\Psi}(\underline{x}, \underline{v}, t) f(\underline{x}, \underline{v}, t) d\underline{v}$$

Or

$$n(\underline{x}, t) \langle \underline{\Psi}(\underline{x}, t) \rangle = \int_{-\infty}^{+\infty} \underline{\Psi}(\underline{x}, \underline{v}, t) f(\underline{x}, \underline{v}, t) d\underline{v} .$$

Consider function

$$\Psi(\underline{x}, \underline{v}, t) = \Psi(\underline{v}) = \underline{v}^{p}$$

We define

The pth moment of Boltzmann equation $= \int_{-\infty}^{+\infty} \Psi$ [Boltzmann equation] $d\underline{v}$
$= \int_{-\infty}^{+\infty} \underline{v}^{p}$ [Boltzmann equation] $d\underline{v}$

Note that it is triple integral over all 3 velocity components here.

Let's look at each term of the Boltzmann equation.

a. Term 1: $\frac{\partial}{\partial t} f(\underline{x}, \underline{v}, t)$

General Moment of Term 1

$$= \int_{-\infty}^{+\infty} \Psi \frac{\partial f}{\partial t} d\underline{v}$$

$$= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \Psi f d\underline{v} \qquad \text{since } \Psi \text{ is not a function of time}$$

$$= \frac{\partial}{\partial t} \left[n \langle \Psi \rangle \right]$$

b. Term 2: $\underline{v} \bullet \underline{\nabla}_{x} f(\underline{x}, \underline{v}, t)$

General Moment of Term 2

$$= \int_{-\infty}^{+\infty} \Psi\left(\underline{v} \bullet \underline{\nabla}_x f\right) d\underline{v}$$

$$= \int_{-\infty}^{+\infty} \underline{\nabla}_x f \bullet \underline{v}\Psi d\underline{v}$$

$$= \underline{\nabla}_x \bullet \int_{-\infty}^{+\infty} f\,\underline{v}\Psi d\underline{v} \qquad \text{since } \Psi \text{ and } \underline{v} \text{ are not functions of } \underline{x}$$

$$= \underline{\nabla}_x \bullet \left[n\left\langle \underline{v}\Psi \right\rangle\right]$$

Note that Ψ is a vector, then $\underline{v}\Psi$ is a rank-2 dyadic function where

$$\underline{v}\Psi = \begin{bmatrix} v_x\Psi_x & v_x\Psi_y & v_x\Psi_z \\ v_y\Psi_x & v_y\Psi_y & v_y\Psi_z \\ v_z\Psi_x & v_z\Psi_y & v_z\Psi_z \end{bmatrix}$$

and divergence of this dyadic function is

$$\begin{aligned} \underline{\nabla}_x \bullet \underline{v}\Psi = &\left[(v_x\Psi_x)\frac{\partial}{\partial x} + (v_y\Psi_x)\frac{\partial}{\partial y} + (v_z\Psi_x)\frac{\partial}{\partial z}\right]\hat{x} \\ &+\left[(v_x\Psi_y)\frac{\partial}{\partial x} + (v_y\Psi_y)\frac{\partial}{\partial y} + (v_z\Psi_y)\frac{\partial}{\partial z}\right]\hat{y} \\ &+\left[(v_x\Psi_z)\frac{\partial}{\partial x} + (v_y\Psi_z)\frac{\partial}{\partial y} + (v_z\Psi_z)\frac{\partial}{\partial z}\right]\hat{z} \end{aligned}$$

c. Term 3: $\frac{q}{m}\left(\underline{E}(\underline{x},t) + \underline{v} \times \underline{B}(\underline{x},t)\right) \bullet \underline{\nabla}_v f(\underline{x},\underline{v},t)$

This term is a bit tricky. We will separate the term $\frac{q}{m}\left(\underline{E}(\underline{x},t) + \underline{v} \times \underline{B}(\underline{x},t)\right)$ into 2 terms: velocity independent and velocity dependent terms.

1. Velocity independent term, $\frac{q}{m}\underline{E}(\underline{x},t)$

   General Moment of Velocity Independent Term 3

$$= \frac{q}{m} \int_{-\infty}^{+\infty} \Psi\left(\underline{E} \bullet \underline{\nabla}_v f\right) d\underline{v}$$

$$= \frac{q}{m} \underline{E} \bullet \int_{-\infty}^{+\infty} \left(\underline{\nabla}_v f\right) \Psi d\underline{v}$$

$$= \frac{q}{m} \underline{E} \bullet \int_{-\infty}^{+\infty} \left[\underline{\nabla}_v (\Psi f) - f \underline{\nabla}_v \Psi\right] d\underline{v}$$

$$= \frac{q}{m} \underline{E} \bullet \left[\int_{-\infty}^{+\infty} \underline{\nabla}_v (\Psi f) d\underline{v} - \int_{-\infty}^{+\infty} f \underline{\nabla}_v \Psi d\underline{v} \right]$$

$$= \frac{q}{m} \underline{E} \bullet \left[\hat{e} \Psi f \Big|_{-\infty}^{+\infty} - n \left\langle \underline{\nabla}_v \Psi \right\rangle \right] \qquad \text{where } \hat{e} = \hat{x} + \hat{y} + \hat{z}$$

Note that $\Psi f \sim v^p \exp\left(-\frac{1}{2} m v^2 \big/ KT \right)$. The exponential term goes to zero faster than the v^p term as $v \to +\infty$ or $v \to -\infty$. Thus, $\Psi f \Big|_{-\infty}^{+\infty} = 0$.

$$= -\frac{qn}{m} \underline{E} \bullet \left\langle \underline{\nabla}_v \Psi \right\rangle$$

2. Velocity dependent term, $\frac{q}{m} \left(\underline{v} \times \underline{B}(\underline{x}, t) \right)$

General Moment of Velocity Dependent Term 3

$$= \frac{q}{m} \int_{-\infty}^{+\infty} \Psi\left((\underline{v} \times \underline{B}) \bullet \underline{\nabla}_v f \right) d\underline{v}$$

$$= \frac{q}{m} \int_{-\infty}^{+\infty} \Psi \sum_i \left[(\underline{v} \times \underline{B})_i \frac{\partial f}{\partial v_i} \right] d\underline{v}$$

Note that $(\underline{v} \times \underline{B})_i$ does not contain v_i. Thus, we can move the term into $\partial / \partial v_i$.

$$= \frac{q}{m} \int_{-\infty}^{+\infty} \Psi \sum_i \left[\frac{\partial}{\partial v_i} \left(f (\underline{v} \times \underline{B})_i \right) \right] d\underline{v}$$

$$= \frac{q}{m} \int_{-\infty}^{+\infty} \sum_i \left[\frac{\partial}{\partial v_i} \left(\Psi f \left(\underline{v} \times \underline{B} \right)_i \right) - f \left(\underline{v} \times \underline{B} \right)_i \frac{\partial \Psi}{\partial v_i} \right] d\underline{v}$$

$$= \frac{q}{m} \left\{ \int_{-\infty}^{+\infty} \sum_i \left[\frac{\partial}{\partial v_i} \left(\Psi f \left(\underline{v} \times \underline{B} \right)_i \right) \right] d\underline{v} - \int_{-\infty}^{+\infty} \sum_i \left[f \left(\underline{v} \times \underline{B} \right)_i \frac{\partial \Psi}{\partial v_i} \right] d\underline{v} \right\}$$

$$= \frac{q}{m} \left\{ \sum_i \left[\int_{-\infty}^{+\infty} \frac{\partial}{\partial v_i} \left(\Psi f \left(\underline{v} \times \underline{B} \right)_i \right) d\underline{v} \right] - \int_{-\infty}^{+\infty} f \left(\left(\underline{v} \times \underline{B} \right) \bullet \underline{\nabla}_v \right) \Psi d\underline{v} \right\}$$

$$= \frac{q}{m} \left\{ \sum_i \left[\int_{-\infty}^{+\infty} \frac{\partial}{\partial v_i} \left(\Psi f \left(\underline{v} \times \underline{B} \right)_i \right) d\underline{v} \right] - n \left\langle \left(\left(\underline{v} \times \underline{B} \right) \bullet \underline{\nabla}_v \right) \Psi \right\rangle \right\}$$

Note that if we integrate the first term with respect to v_i first, we can move $\left(\underline{v} \times \underline{B} \right)_i$ out of the integral. Then we have $\partial \left(\Psi f \left(\underline{v} \times \underline{B} \right)_i \right) / \partial v_i$ left inside the integral. Since we integrate with respect to v_i, we will then get $\Psi f \Big|_{-\infty}^{+\infty}$ which is zero. Thus, the first term above is zero.

$$= -\frac{qn}{m} \left\langle \left(\left(\underline{v} \times \underline{B} \right) \bullet \underline{\nabla}_v \right) \Psi \right\rangle$$

General Moment of Term 3 $= -\dfrac{qn}{m} \left[\underline{E} \bullet \left\langle \underline{\nabla}_v \Psi \right\rangle + \left\langle \left(\left(\underline{v} \times \underline{B} \right) \bullet \underline{\nabla}_v \right) \Psi \right\rangle \right]$

d. Term 4: $\left. \dfrac{\partial f}{\partial t} \right|_{\text{collision}}$

General Moment of Term 4

$$= \int_{-\infty}^{+\infty} \Psi \left. \frac{\partial f}{\partial t} \right|_{\text{collision}} d\underline{v}$$

$$= \left(\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \Psi f d\underline{v} \right)_{\text{collision}}$$

$$= \left(\frac{\partial}{\partial t} \left[n \left\langle \Psi \right\rangle \right] \right)_{\text{collision}}$$

Thus, general moment of Boltzmann equation is

$$\boxed{\frac{\partial}{\partial t}\left[n\langle\Psi\rangle\right]+\underline{\nabla}_x \bullet \left[n\langle\Psi\underline{v}\rangle\right]-\frac{qn}{m}\left[\underline{E}\bullet\langle\underline{\nabla}_v\Psi\rangle+\langle(\underline{v}\times\underline{B})\bullet\underline{\nabla}_v\Psi\rangle\right]=\begin{cases}\left(\frac{\partial}{\partial t}\left[n\langle\Psi\rangle\right]\right)_{\text{collision}}\\ 0\end{cases}}$$

How do we use this moment equation?

What this moment equation gives us is transfer functions for various quantities:

p = 0 gives transfer function of mass, i.e. continuity equation

p = 1 gives transfer function of momentum, i.e. $\sim mv$

p = 2 gives transfer function of energy, i.e. $\sim \frac{1}{2}mv^2$

Let's look at each case.

1. $p=0$, $\Psi = v^0 = 1$.

Thus, $\langle\Psi\rangle = 1$. The moment equation becomes

$$\frac{\partial n}{\partial t}+\underline{\nabla}_x \bullet \left[n\langle\underline{v}\rangle\right]-\frac{qn}{m}\left[\underline{E}\bullet\langle\underline{\nabla}_v 1\rangle+\langle(\underline{v}\times\underline{B})\bullet\underline{\nabla}_v 1\rangle\right]=\begin{cases}\left(\frac{\partial n}{\partial t}\right)_{\text{collision}}\\ 0\end{cases}$$

$$\frac{\partial n}{\partial t}+\underline{\nabla}_x \bullet \left[n\underline{u}\right]=\begin{cases}\left(\frac{\partial n}{\partial t}\right)_{\text{collision}}\\ 0\end{cases}$$ since $\underline{u} = \langle\underline{v}\rangle =$ fluid velocity

which is the continuity equation shown before in the last lecture.

2. $p=1$, $\Psi = m\underline{v}^1 = m\underline{v}$.

Thus, $\langle\Psi\rangle = m\underline{u}$. The moment equation becomes

$$\frac{\partial}{\partial t}\left[nm\underline{u}\right]+\underline{\nabla}_x \bullet \left[n\langle m\underline{v}\underline{v}\rangle\right]-\frac{qn}{m}\left[\underline{E}\bullet\langle\underline{\nabla}_v m\underline{v}\rangle+\langle(\underline{v}\times\underline{B})\bullet\underline{\nabla}_v m\underline{v}\rangle\right]=\begin{cases}\left(\frac{\partial}{\partial t}\left[nm\underline{u}\right]\right)_{\text{collision}}\\ 0\end{cases}$$

It is easy to see that $\langle\underline{\nabla}_v\underline{v}\rangle = \overleftrightarrow{I}$ where $\overleftrightarrow{I}$ is unit tensor so that

$$\frac{qn}{m}\left[\underline{E}\bullet\left\langle\underline{\nabla}_v m\underline{v}\right\rangle+\left\langle\left(\underline{v}\times\underline{B}\right)\bullet\underline{\nabla}_v m\underline{v}\right\rangle\right]=qn\left[\underline{E}+\underline{u}\times\underline{B}\right]$$

However, what is $\left\langle\underline{v}\underline{v}\right\rangle$?

As we have seen before, $\underline{v}\underline{v}$ is a dyadic function where

$$\underline{v}\underline{v}=\begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}$$

In an isotropic system, all the off-diagonal components go to zero when we take average, i.e. we assume first that $\left\langle\underline{v}\right\rangle=\underline{u}=0$. Also, $\left\langle v_x v_x\right\rangle=\left\langle v_y v_y\right\rangle=\left\langle v_z v_z\right\rangle=v_{thermal}^2$. Thus,

$$\left\langle\underline{v}\underline{v}\right\rangle=v_{thermal}^2\overleftrightarrow{I}\ .$$

$\underline{\nabla}_x\bullet\left[n\left\langle m\underline{v}\underline{v}\right\rangle\right]$ gives $\underline{\nabla}_x\bullet nmv_{thermal}^2\overleftrightarrow{I}$ which is essentially pressure gradient force $\underline{\nabla}p$.

If, however, there is net velocity $\underline{u}$, i.e. the isotropic system moves at fluid velocity $\underline{u}$, then $f\left(\underline{x},t\right)=n\left(\underline{x},t\right)\delta\left(\underline{v}-\underline{u}\right)$

$$\begin{aligned}\left\langle\underline{v}\underline{v}\right\rangle&=\frac{1}{n}\int\underline{v}\underline{v}fd\underline{v}\\&=\frac{1}{n}\int\underline{v}\underline{v}n\delta\left(\underline{v}-\underline{u}\right)d\underline{v}\\&=\underline{u}\underline{u}\end{aligned}$$

Thus, total $\left\langle\underline{v}\underline{v}\right\rangle$ is

$$\left\langle\underline{v}\underline{v}\right\rangle=\underline{u}\underline{u}+v_{thermal}^2\overleftrightarrow{I}$$

Consequently, we have

$$\frac{\partial}{\partial t}\left[nm\underline{u}\right]+\underline{\nabla}_x\bullet nm\underline{u}\underline{u}+\underline{\nabla}p-qn\left[\underline{E}+\underline{u}\times\underline{B}\right]=\begin{cases}\left(\dfrac{\partial}{\partial t}\left[nm\underline{u}\right]\right)_{\text{collision}}\\0\end{cases}$$

As a homework, show that this is the same as fluid equation given in the last lecture!!!

3. $p = 2$, $\Psi = \frac{1}{2} m \underline{v}^2 = \frac{1}{2} m \underline{v}\underline{v}$.

For simplicity, we will assume that we have isotropic system.

$$\frac{\partial}{\partial t}\left[n\left\langle \frac{1}{2} m \underline{v}\underline{v} \right\rangle \right] + \underline{\nabla}_x \bullet \left[n\left\langle \frac{1}{2} m \underline{v}\underline{v}\underline{v} \right\rangle \right] - \frac{qn}{m}\left[\underline{E} \bullet \left\langle \underline{\nabla}_v \frac{1}{2} m \underline{v}\underline{v} \right\rangle + \left\langle (\underline{v} \times \underline{B}) \bullet \underline{\nabla}_v \frac{1}{2} m \underline{v}\underline{v} \right\rangle \right]$$

$$= \begin{cases} \left(\frac{\partial}{\partial t}\left[n\left\langle \frac{1}{2} m \underline{v}\underline{v} \right\rangle \right] \right)_{\text{collision}} \\ 0 \end{cases}$$

Let's look at each term,

$\langle \underline{v}\underline{v} \rangle = \underline{u}\underline{u} + v_{thermal}^2 \overleftrightarrow{I}$ as we have before for isotropic system moving at fluid speed $\underline{u}$.

$\langle \underline{v}\underline{v}\underline{v} \rangle = \left\langle \sum_{i,j,k} v_i v_j v_k \hat{i}\hat{j}\hat{k} \right\rangle$ where $\hat{i}, \hat{j}, \hat{k}$ are combinations of $\hat{x}, \hat{y}, \hat{z}$.

Again, assume isotropic system moving at speed $\underline{u}$, then $\left\langle v_i v_j v_k \hat{i}\hat{j}\hat{k} \right\rangle$ is only non-zero when *i* = *j* = *k*.

It follows that $\langle \underline{v}\underline{v}\underline{v} \rangle = \underline{u}\underline{u}\underline{u} + v_{thermal}^3 \overleftrightarrow{\overleftrightarrow{I}}$

$\langle \underline{\nabla}_v \underline{v}\underline{v} \rangle = 2\underline{u}\overleftrightarrow{I}$

$\langle (\underline{v} \times \underline{B}) \bullet \underline{\nabla}_v \underline{v}\underline{v} \rangle = 0$ since the two terms are perpendicular to each other.

We then get,

$$\frac{\partial}{\partial t}\left[\frac{1}{2} nm\left(\underline{u}\underline{u} + v_{thermal}^2 \overleftrightarrow{I} \right) \right] + \underline{\nabla}_x \bullet \left[\frac{1}{2} nm\left(\underline{u}\underline{u}\underline{u} + v_{thermal}^3 \overleftrightarrow{\overleftrightarrow{I}} \right) \right] - \frac{qn}{m}\left[\underline{E} \bullet m\underline{u}\overleftrightarrow{I} \right]$$

$$= \begin{cases} \left(\frac{\partial}{\partial t}\left[n\left\langle \frac{1}{2} m \underline{v}\underline{v} \right\rangle \right] \right)_{\text{collision}} \\ 0 \end{cases}$$

In general if we do not assume isotropic condition, the energy transfer reads

$$\frac{\partial}{\partial t}\left[\frac{1}{2}nm\underline{u}\underline{u}+\frac{3}{2}nKT\overleftrightarrow{I}\right]+\underline{\nabla}_x\bullet\left[\frac{1}{2}nm\underline{u}\underline{u}\underline{u}+\frac{3}{2}nKT\underline{u}\overleftrightarrow{I}\right]+Q+\underline{P}\bullet\underline{u}\overleftrightarrow{I}-\underline{E}\bullet\underline{j}\overleftrightarrow{I}=0$$

where the first parenthesis is the energy density unit,

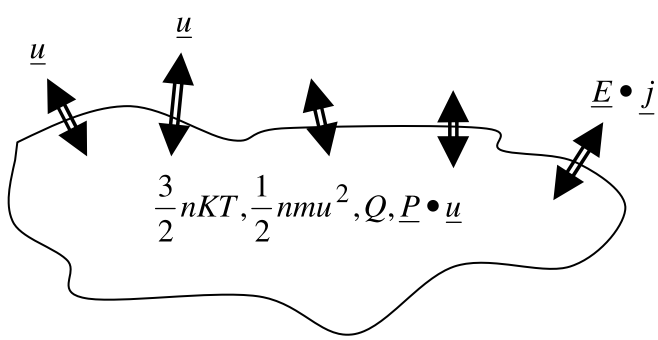
the second parenthesis is the energy flux,

Q is the heat flux,

$\underline{P}\bullet\underline{u}\overleftrightarrow{I}$ is the mechanical work, and

$\underline{E}\bullet\underline{j}\overleftrightarrow{I}$ is the electric work, also commonly known as Ohmic heating.

What this shows us is how energy move around in a plasma fluid system.



Plasma approximation

Before we go further, there is one important approximation of plasma fluid that you will hear often.

We have discussed before about quasineutrality of plasma. That is, plasma has tendency to be neutral.

From now on, we will get into a topic of waves in plasma. The following assumption is useful:

For low frequency plasma, we can normally set $n_e = n_i$ because the time scale is long can ion can catch up with electron. $\underline{\nabla}\bullet\underline{E}$ however is not zero, and E field can be found from equation of motion.

For high frequency plasma, we cannot set $n_e = n_i$, and E field has to be calculated from Maxwell's equation