Last Time

Moments of Vlasov equation

- Zeroth moment = mas transfer equation (continuity equation)
- First moment = momentum transfer equation (fluid equation)
- Second moment = energy transfer equation

Plasma approximation

- For low frequency plasma, we can normally set $n_e = n_i$ because the time scale is long can ion can catch up with electron. $E \bullet$ □ however is not zero, and E field can be found from equation of motion.
- For high frequency plasma, we cannot set $n_e = n_i$, and E field has to be calculated from Maxwell's equation

Waves in Plasmas

We have discussed about plasma oscillation previously. The momentary imbalance of charges and the tendency of plasma to restore its charge neutrality state cause plasma to oscillate at “plasma frequency” ω_p.

Nevertheless, I only introduced electron plasma frequency, ω_{pe}. In reality, however, each species of ions also oscillates at its corresponding ion plasma frequency, ω_{pi}. Since ions are much heavier than electron in general, ω_{pi} is usually much smaller than ω_{pe}. So it is electron that does most of the oscillation.

Another frequency which we have discussed is the cyclotron frequency, ω_c. This frequency only occurs in the presence of magnetic field. Each ion species and electron in plasma has its corresponding cyclotron frequency which depends on its charge. Thus, you will also see differentiation between electron and electron cyclotron frequencies, ω_{ce} and ω_{ci}.

These, however, are not the only types of oscillations which can exist in plasmas as we shall see. Plasma has ability to propagate waves from other sources. It is these propagation of waves that allows energy to transfer inside plasma. Some waves can grow and become unstable. Some waves get reflected by plasma. Some get absorbed.

Wave representation

To represent wave in plasma, one usually use the following notation:

$$A = A_0 e^{i\left(\underline{k}\bullet\underline{r}-\omega t\right)}$$

where A is a wave quantity, A_0 is amplitude of the wave, $e^{i\underline{k}\bullet\underline{r}}$ represents wave propagation, and $e^{-i\omega t}$ represents wave oscillation. In Cartesian coordinate, $\underline{k}\bullet\underline{r} = k_x x + k_y y + k_z z$.

As in any periodic motion, there can exist more than one frequency in plasma. Fourier analysis can be used to separate those frequencies.

$$f(t) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t} f(\omega)d\omega \Leftrightarrow f(\omega) = \frac{1}{\sqrt{2\pi}}\int e^{i\omega t} f(t)dt$$

Phase vs. group velocities

The argument $\underline{k}\bullet\underline{r}-\omega t$ in the wave representation is known as “phase”. Phase of wave moves at a velocity which can be calculate by setting

$$\frac{d}{dt}(\underline{k}\bullet\underline{r}-\omega t)=0$$

If wave propagates in x direction, for instance, then

$$\frac{d}{dt}(kx-\omega t)=0 \Rightarrow k\frac{dx}{dt}-\omega\frac{dt}{dt}=0$$

$$\boxed{v_p \equiv \frac{dx}{dt}=\frac{\omega}{k}}$$

v_p is known as ***phase velocity***. Since phase is not the quantity of wave that carries information, it does not have to obey theory of relativity, and can move at speed greater than speed of light.

Information gets carried in a wave by wave package, which is in the form of beating (modulation) among waves. Consider 2 waves,

$$A_1 = A_0 e^{i((k+\Delta k)x-(\omega+\Delta\omega)t)} \text{ and } A_2 = A_0 e^{i((k-\Delta k)x-(\omega-\Delta\omega)t)}$$

Beating between these 2 waves is

$$A_1 + A_2 = A_0\left[e^{i(\Delta kx-\Delta\omega t)}+e^{-i(\Delta kx-\Delta\omega t)}\right]e^{i(kx-\omega t)}$$
$$=2A_0\cos(\Delta kx-\Delta\omega t)e^{i(kx-\omega t)}$$

We can see that the wave package still propagates at phase velocity, but the content inside the package now moves at $\Delta\omega/\Delta k$. If we take limit so that $\Delta \to 0$, then

$$\boxed{v_g \equiv \frac{d\omega}{dk}}$$

v_g is known as ***group velocity***. This velocity must obey theory of relativity, and thus cannot be greater than speed of light.

Process of Linearization

For wave quantity, the spatial dependence is in the $e^{i\underline{k}\bullet\underline{r}}$ term, while the time dependence is in the $e^{-i\omega t}$ term. Thus, when we encounter space and time derivatives, we can replace them by

$$\frac{dA}{dx}=ik_xA \text{ and } \frac{dA}{dt}=-i\omega A$$

In addition, when there are slowly varying and fast varying terms in the same equation, they can usually be separated as will be made clear below.

This is called process of linearization.

Electrostatic waves

The word "electrostatic" means that there is no perturbation in B field, i.e. $B_1 = 0$.

From Faraday's law,

$$\underline{\nabla} \times \underline{E}_1 = -\frac{d\underline{B}_1}{dt}.$$

Then, we can conclude that $\underline{k} \parallel \underline{E}_1$, or wave propagates in the same direction as E_1 field.

Recall Vlasov equation,

$$\frac{\partial}{\partial t}(f_0 + f_1) + \underline{v} \bullet \underline{\nabla}_x (f_0 + f_1) + \frac{q}{m}\left((\underline{E}_0 + \underline{E}_1) + \underline{v} \times (\underline{B}_0 + \underline{B}_1)\right) \bullet \underline{\nabla}_v (f_0 + f_1) = 0$$

where subscripts 0 and 1 represent equilibrium (or slowly varying) and perturbation (or fast varying) terms respectively. Since we are only interested in effect of the perturbed E and B fields, and $\underline{k} \parallel \underline{E}$, we can let $E_0 = B_0 = 0$. If we let $\underline{E} = E_1 \hat{x}$, then we can rewrite Vlasov equation as

$$\frac{\partial}{\partial t}(f_0 + f_1) + v\frac{\partial}{\partial x}(f_0 + f_1) + \frac{q}{m}E_1\frac{\partial}{\partial v}(f_0 + f_1) = 0$$

We shall now try to linearize this equation.

Notice that when there is no perturbation,

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = 0$$

since they are equilibrium terms (zeroth order terms). The term $E_1 \frac{\partial f_1}{\partial v}$, which is second order term, is small in comparison to the terms $\frac{\partial f_1}{\partial t}$ and $\frac{\partial f_1}{\partial x}$, which are first order terms. Thus, we are left with

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{q}{m}E_1\frac{\partial f_0}{\partial v} = 0$$

$$-i\omega f_1 + ikvf_1 + \frac{q}{m}E_1\frac{\partial f_0}{\partial v} = 0$$

Another equation between E and *f* is needed to close solve this problem. We can use Gauss's law from Maxwell's equations

$$\frac{\partial E_1}{\partial x} = \sum_i \frac{q_i}{\varepsilon_0}(n_{i0} + n_{i1}) = \sum_i \frac{q_i}{\varepsilon_0} \int (f_{i0} + f_{i1}) dv$$

where we use ε_0 for free-space permittivity. Linearization gives

$$ikE_1 = \sum_i \frac{q_i}{\varepsilon_0} \int f_{i1} dv$$

For electron electrostatic wave (or high frequency electrostatic wave), we then have a system of 2 equations:

$$-i\omega f_1 + ikvf_1 - \frac{e}{m} E_1 \frac{\partial f_0}{\partial v} = 0$$

$$ikE_1 = -\frac{e}{\varepsilon_0} \int f_1 dv$$

Solving the first equation for f_1 gives

$$f_1 = i\frac{e}{m} E_1 \frac{(\partial f_0/\partial v)}{(\omega - kv)} = -i\frac{e}{m} E_1 \frac{(\partial f_0/\partial v)}{k(v - \omega/k)}$$

Substitute this in the second equation and write $f_0 = n_0 g(v)$ yields

$$\begin{aligned} ikE_1 &= -\frac{e}{\varepsilon_0} \int -i\frac{e}{m} E_1 \frac{n_0(\partial g(v)/\partial v)}{k(v - \omega/k)} dv \\ &= i\frac{e^2 n_0}{\varepsilon_0 m} \frac{E_1}{k} \int \frac{(\partial g(v)/\partial v)}{(v - \omega/k)} dv \\ &= i\frac{\omega_{pe}^2}{k} E_1 \int \frac{(\partial g(v)/\partial v)}{(v - \omega/k)} dv \end{aligned}$$

$$\begin{aligned} 0 &= ikE_1 - i\frac{\omega_{pe}^2}{k} E_1 \int \frac{(\partial g(v)/\partial v)}{(v - \omega/k)} dv \\ &= ik\left[1 - \frac{\omega_{pe}^2}{k^2} \int \frac{(\partial g(v)/\partial v)}{(v - \omega/k)} dv\right] E_1 \end{aligned}$$

If we view plasma as dielectric medium and there is no free charge in it, this is exactly the same as writing the following Gauss's law

$$\underline{\nabla} \bullet \underline{D} = ik\varepsilon E_1 = 0$$

where properties of dielectric medium are incorporated into ε.

Equating the two equations above, we have

$$\boxed{\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{(\partial g(v)/\partial v)}{(v-\omega/k)}dv}$$

This is called ***dielectric function*** or ***electrostatic plasma dielectric***.

We can get back

$$1-\frac{\omega_{pe}^2}{k^2}\int\frac{(\partial g(v)/\partial v)}{(v-\omega/k)}dv=0$$

by setting $\varepsilon(\omega,k)$ to zero. This $\varepsilon(\omega,k)=0$ equation is called ***dispersion relation*** which shows relationship between ω and k.

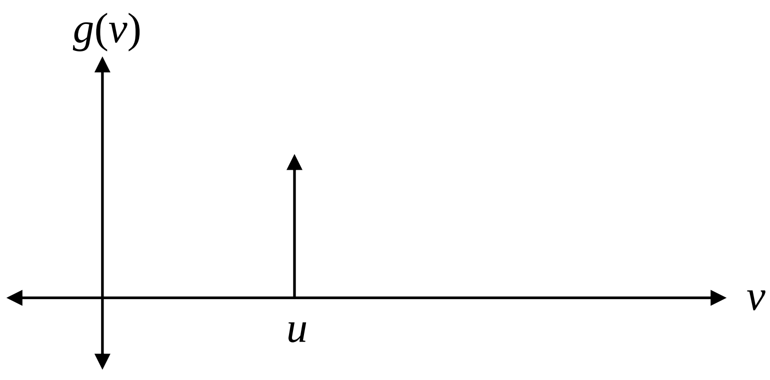
With some mathematical manipulation, the dielectric function can be rewritten

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\left\{\left.\frac{g(v)}{v-\omega/k}\right|_{v=-\infty}^{v=+\infty}+\int_{-\infty}^{+\infty}\frac{g(v)}{(v-\omega/k)^2}dv\right\}$$

$$\boxed{\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{g(v)}{(v-\omega/k)^2}dv}$$

For now, we will assume that $\omega/k>>v$ (high frequency) so that we can avoid zero "pole" from integration. So, only electrons will participate, and we can leave ions out of our calculations.

<u>*Example 1*</u> Consider cold plasma with $g(v)=\delta(v-u)$. That is, all electrons in the plasma have the same velocity *u*.



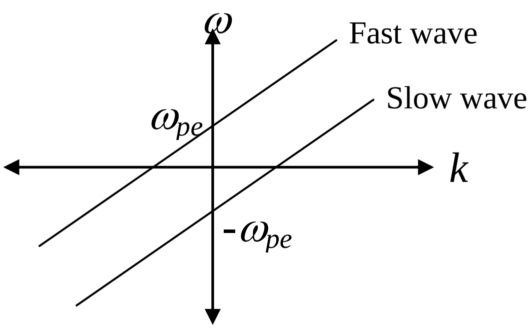
Substitute *g*(*v*) into the dielectric function, we get

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{\delta(v-u)}{(v-\omega/k)^2}dv$$
$$= 1 - \frac{\omega_{pe}^2}{k^2}\left(\frac{1}{u-\omega/k}\right)^2$$
$$= 1 - \frac{\omega_{pe}^2}{(\omega-ku)^2}$$

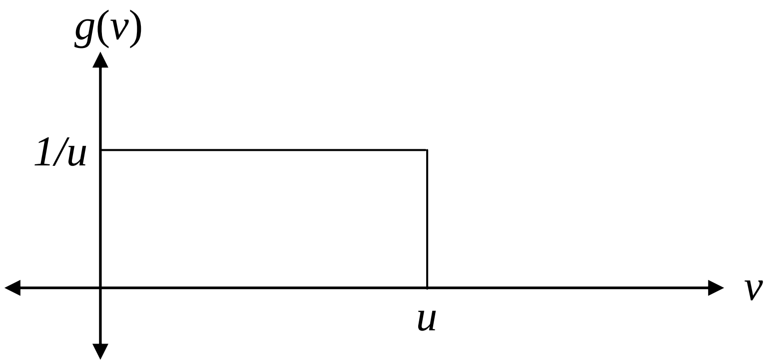
Thus, the dispersion relation for this wave is acquired by setting the LHS to zero, hence

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{(\omega-ku)^2} = 0$$
$$(\omega-ku)^2 = \omega_{pe}^2$$
$$\omega - ku = \pm\omega_{pe}$$
$$\boxed{\omega = ku \pm \omega_{pe}}$$

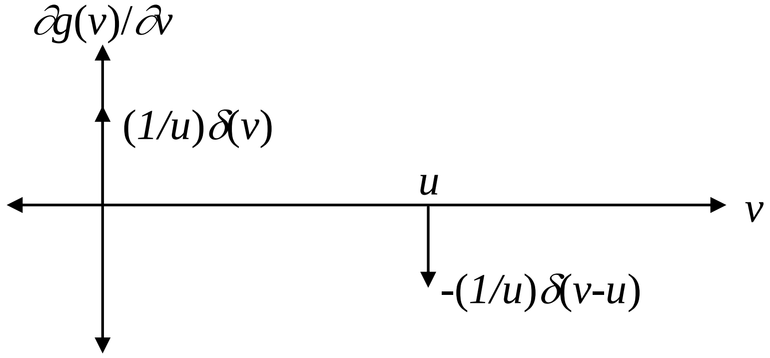
Notice that there are 2 resulting waves here. One is slow wave with $\omega = ku - \omega_{pe}$, i.e. slower than electron velocity, and another is fast wave with $\omega = ku + \omega_{pe}$, i.e. faster than electron velocity. We can plot this dispersion relation in **dispersion diagram**.



Example 2 $g(v)$ has the following profile.



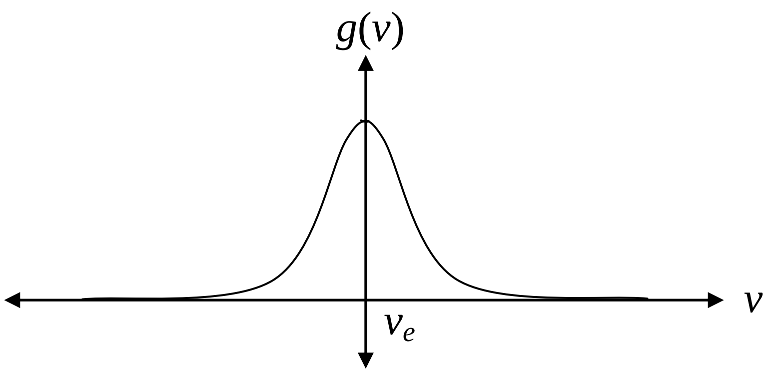
It is easier to evaluate $\varepsilon(\omega,k)$ by using the first form of the dielectric function. We can replace the above profile with impulse profile, $\partial g(v)/\partial v$.



Thus, if we substitute $\partial g(v)/\partial v$ into the first form of the dielectric function, we get,

$$\begin{aligned}\varepsilon(\omega,k) &= 1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{\left((1/u)\delta(v)-(1/u)\delta(v-u)\right)}{(v-\omega/k)}dv \\ &= 1-\frac{\omega_{pe}^2}{k^2}\left(\frac{(1/u)}{-\omega/k}-\frac{(1/u)}{u-\omega/k}\right) \\ &= 1+\frac{\omega_{pe}^2}{\omega ku}-\frac{\omega_{pe}^2}{\omega ku-k^2u^2}\end{aligned}$$

Example 3 Warm plasma has Maxwellian distribution instead of delta function.



Using the second form of the dielectric function,

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{g(v)}{(v-\omega/k)^2}dv$$

Because $\omega/k >> v$, we can expand the denominator inside the integral

$$\frac{1}{(v-\omega/k)^2}=\frac{1}{(\omega/k)^2(1-kv/\omega)^2}$$
$$=\left(\frac{k}{\omega}\right)^2\left(1+2\left(\frac{kv}{\omega}\right)+3\left(\frac{kv}{\omega}\right)^2+\dots\right)$$

Thus,

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}g(v)\left(\frac{k}{\omega}\right)^2\left(1+2\left(\frac{kv}{\omega}\right)+3\left(\frac{kv}{\omega}\right)^2+\dots\right)dv$$
$$=1-\frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{+\infty}g(v)\left(1+2\left(\frac{kv}{\omega}\right)+3\left(\frac{kv}{\omega}\right)^2+\dots\right)dv$$

Notice that $\int_{-\infty}^{+\infty}g(v)dv=1$, $\int_{-\infty}^{+\infty}g(v)v\,dv=0$, and $\int_{-\infty}^{+\infty}g(v)v^2dv=v_{th}^2/2=v_e^2$. Leaving only up to third term (because the terms after that are small), we then get

$$\varepsilon(\omega,k)=1-\frac{\omega_{pe}^2}{\omega^2}\left(1+3\frac{k^2v_e^2}{\omega^2}\right)$$

The dispersion relation is then, by setting $\varepsilon(\omega,k)=0$,

$$1-\frac{\omega_{pe}^2}{\omega^2}\left(1+3\frac{k^2v_e^2}{\omega^2}\right)=0$$
$$\frac{\omega_{pe}^2}{\omega^2}\left(1+3\frac{k^2v_e^2}{\omega^2}\right)=1$$
$$\omega_{pe}^2+3\left(\frac{\omega_{pe}^2}{\omega^2}\right)k^2v_e^2=\omega^2$$

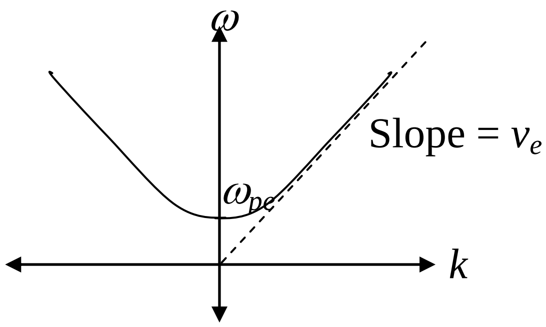
To the lowest order, $\omega^2\approx\omega_{pe}^2$. Then,

$$\boxed{\omega^2=\omega_{pe}^2+3k^2v_e^2}$$

This is known as **Bohm-Gross dispersion** relation. It also has other names, e.g. **electron plasma wave** and **Langmuir wave**.

We can view this as propagation of plasma oscillation. Because of the oscillation, alternating electric field is set up, and electrons move along the field at speed v_e which is $v_{th}/\sqrt{2}$.

The dispersion diagram is as shown below.



Notice that if $\omega^2 < \omega_{pe}^2$, then $3k^2 v_e^2$ would be negative, and thus k is imaginary indicating that wave (plasma oscillation) cannot propagating through plasma.

Example 4 Now let's look at the low frequency case so that ions can respond. By low frequency, we mean $\omega << \omega_{pi}$. The dispersion relation then becomes

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{k^2}\int_{-\infty}^{+\infty}\frac{(\partial g_e(v)/\partial v)}{(v-\omega/k)}dv - \frac{\omega_{pi}^2}{k^2}\int_{-\infty}^{+\infty}\frac{(\partial g_i(v)/\partial v)}{(v-\omega/k)}dv$$

Let assume that $kv_i << \omega << kv_e$ (we will relax this assumption later). We then get

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{\omega^2 - k^2 v_e^2} - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_i^2}$$

Note that without the ion term, this dispersion relation is equivalent to the Langmuir dispersion relation above.

Since $kv_i << \omega << kv_e$,

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{-k^2 v_e^2} - \frac{\omega_{pi}^2}{\omega^2}$$

Set $\varepsilon(\omega,k) = 0$,

$$0 = 1 - \frac{\omega_{pe}^2}{-k^2 v_e^2} - \frac{\omega_{pi}^2}{\omega^2}$$

$$\omega^2 = k^2 v_e^2\left(\frac{\omega_{pi}^2}{\omega_{pe}^2}\right) = k^2 v_e^2 \frac{m_e}{m_i} = k^2 \frac{T_e}{m}$$

$$\boxed{\frac{\omega}{k} = \sqrt{\frac{T_e}{m_i}}}$$

This is known as **ion acoustic dispersion relation**.

At low frequency, movement of ions governs the oscillation. Electrons follow ions to maintain charge neutrality. Electrons themselves appear isothermal since they move much faster.

We will come back to this dielectric function again when we discuss about instability.

For study of other waves in plasmas, we shall go back to direct fluid approach.

1. Electron plasma wave (Langmuir wave)

   We have already discussed about this wave. So I will skip this one.

2. Ion plasma wave

   The ion acoustic wave that we have discussed earlier was arrived at assumption that $kv_i << \omega << kv_e$. We shall now relax this assumption.

   Writing continuity equations, fluid equations, and Gauss's law for ions and electrons:

$$\begin{aligned}
&\partial_t n_e + \partial_x(n_e v_e) = 0 \\
&m_e n_e \partial_t v_e + m_e n_e v_e \partial_x v_e = -\partial_x p_e - en_e E \\
&\partial_t n_i + \partial_x(n_i v_i) = 0 \\
&m_i n_i \partial_t v_i + m_i n_i v_i \partial_x v_i = -\partial_x p_i + en_i E \\
&\partial_x E = e(n_i - n_e)/\varepsilon_0
\end{aligned}$$

   Linearization yields

$$\begin{aligned}
&-i\omega n_{e1} + ikn_0 v_{e1} = 0 \\
&-i\omega m_e n_0 v_{e1} = -ikp_{e1} - en_0 E_1 \\
&-i\omega n_{i1} + ikn_0 v_{i1} = 0 \\
&-i\omega m_i n_0 v_{i1} = -ikp_{i1} + en_0 E_1 \\
&ikE_1 = e(n_{i1} - n_{e1})/\varepsilon_0
\end{aligned}$$

   Because m_e is very small, we can ignore the LHS of the second equation. Write, $p_{e1} = \gamma_e n_{e1} T_e$ and $p_{i1} = \gamma_i n_{i1} T_i$ then the second and the forth equations become

$$ikn_{e1} = -\frac{en_0}{\gamma_e T_e} E_1$$

$$-i\omega m_i n_0 v_{i1} = -ik\gamma_i T_i n_{i1} + en_0 E_1$$

   From the linearized continuity equation of ions, we can eliminate $n_0 v_{i1}$ from the second equation, which then becomes

$$-i\omega m_i \frac{\omega}{k} n_{i1} = -ik\gamma_i T_i n_{i1} + en_0 E_1$$

$$\left(\omega^2 - k^2 \frac{\gamma_i T_i}{m_i}\right) n_{i1} = ik \frac{en_0}{m_i} E_1$$

So now we have

$$ikn_{e1} = -\frac{en_0}{\gamma_e T_e} E_1$$

$$\left(\omega^2 - k^2 \frac{\gamma_i T_i}{m_i}\right) n_{i1} = ik \frac{en_0}{m_i} E_1$$

Substitute these 2 equations into the Gauss's law, we get

$$ikE_1 = \frac{e}{\varepsilon_0}\left(\frac{ik \frac{en_0}{m_i}}{\omega^2 - k^2 \frac{\gamma_i T_i}{m_i}} + \frac{\frac{en_0}{\gamma_e T_e}}{ik} \right) E_1$$

$$0 = ikE_1 \left(1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 \frac{\gamma_i T_i}{m_i}} + \frac{\omega_{pe}^2}{k^2 \frac{\gamma_e T_e}{m_e}} \right)$$

The dispersion relation of ion plasma wave is then

$$\boxed{\omega^2 = k^2 \frac{\gamma_i T_i}{m_i} + \frac{k^2 \gamma_e T_e / m_i}{1 + \gamma_e k^2 \lambda_e^2}}$$

where recall $\lambda_e = \sqrt{\frac{\varepsilon_0 T_e}{e^2 n_0}}$

For small $k\lambda_e$ and $T_i \to 0$, we recover the ion acoustic wave.

For large $k\lambda_e$ and $T_i \to 0$, $\omega^2 = \omega_{pi}^2$

This if we plot dispersion diagram of ion plasma wave, we get