Last Time

- Wave representation: $A = A_0 e^{i(\underline{k} \bullet \underline{r} - \omega t)}$
- Phase vs. group velocities: $v_p \equiv \frac{\omega}{k},\ v_g \equiv \frac{d\omega}{dk}$
- Linearization process: $\underline{\nabla} = i\underline{k}$ and $\partial_t = -i\omega$
- Dispersion relation: $\omega - k$ relation
- Dielectric function for electrostatic wave:

$$\varepsilon(\omega,k) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} \frac{(\partial g(v)/\partial v)}{(v - \omega/k)} dv = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} \frac{g(v)}{(v - \omega/k)^2} dv$$

- Cold electron plasma dispersion relation: $\omega = ku \pm \omega_{pe}$
- Electron plasma wave/Langmuir wave dispersion relation: $\omega^2 = \omega_{pe}^2 + 3k^2 v_e^2$
- Ion acoustic dispersion relation: $\frac{\omega}{k} = \sqrt{\frac{T_e}{m_i}}$
- Ion plasma wave dispersion relation: $\omega^2 = k^2 \frac{\gamma_i T_i}{m_i} + \frac{k^2 \gamma_e T_e / m_i}{1 + \gamma_e k^2 \lambda_e^2}$

More on waves in plasmas

3. Electromagnetic waves

These waves are

- high frequency: $\omega \geq \omega_{pe}$ (no need to consider ion motion).
- transverse waves: $\underline{k} \bullet \underline{E}_1 = 0$. (waves travel in the direction perpendicular to electric field) It follows from Faraday's law, $\underline{\nabla} \times \underline{E} = -\partial_t \underline{B}$, that $\underline{k} \bullet \underline{B}_1 = 0$.
- unmagnetized: $B_0 = 0$.

Equations which we need to evaluate these waves are

$$\underline{\nabla} \times \underline{E} = -\partial_t \underline{B}$$

$$\underline{\nabla} \times \underline{H} = \underline{j} + \partial_t \underline{D}$$

$$\underline{j}_e = -en_e \underline{v}_e$$

$$m_e n_e \partial_t \underline{v}_e + m_e n_e (\underline{v}_e \bullet \underline{\nabla}) \underline{v}_e = -\underline{\nabla} p_e - en_e \underline{E} - en_e \underline{v}_e \times \underline{B}$$

$$\partial_t n_e + \underline{\nabla} \bullet n_e \underline{v}_e = 0$$

Notice that we cannot use Gauss's laws here because $\underline{k} \bullet \underline{E} = 0$ and $\underline{k} \bullet \underline{B} = 0$.

There are several assumptions:

- v_e has no zero component.
- $\underline{v}_e \times \underline{B}$ can be neglected in the fluid equation. This assumption will be justified later.

We are interested in solution where $\underline{k} \bullet \underline{v}_e = 0$. So,

- $\partial_t n_e + \underline{\nabla} \bullet n_e \underline{v}_e = \partial_t n_e = 0$ and consequently $\underline{\nabla} p_e = 0$.
- $m_e n_e (\underline{v}_e \bullet \underline{\nabla}) \underline{v}_e = 0$.

We then have

$$\underline{\nabla} \times \underline{E} = -\partial_t \underline{B}$$

$$\underline{\nabla} \times \underline{H} = -en_e \underline{v}_e + \partial_t \underline{D}$$

$$m_e n_e \partial_t \underline{v}_e = -en_e \underline{E}$$

Differentiating the second equation, we get

$$\underline{\nabla} \times \frac{\partial \underline{H}}{\partial t} = -en_e \frac{\partial \underline{v}_e}{\partial t} + \frac{\partial^2 \underline{D}}{\partial t^2}$$

Since $\underline{B} = \mu_0 \underline{H}$, then $\underline{\nabla} \times \underline{E} = -\partial_t \underline{B} = -\mu \partial_t \underline{H}$.

Also, $m_e n_e \partial_t \underline{v}_e = -en_e \underline{E} \rightarrow \partial_t \underline{v}_e = -\frac{e}{m_e} \underline{E}$

Substitute these into the equation above, we get

$$-\underline{\nabla} \times \underline{\nabla} \times \underline{E} = -\mu_0 en_e \left(-\frac{e}{m_e} \underline{E} \right) + \mu_0 \frac{\partial^2 \underline{D}}{\partial t^2} = \mu_0 \frac{e^2 n_e}{m_e} \underline{E} + \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

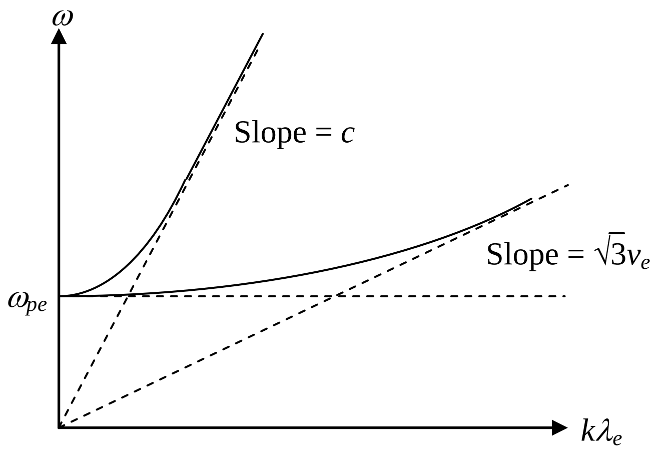
Using identity $\mu_0 \varepsilon_0 = 1/c^2$, and assume plane wave propagation so that $\underline{\nabla} \times \underline{\nabla} \times = -k^2$ and $\partial^2/\partial t^2 = -\omega^2$, then

$$k^2 \underline{E} = \frac{1}{c^2} \left(\frac{e^2 n_e}{m_e \varepsilon_0} \right) \underline{E} - \frac{1}{c^2} \omega^2 \underline{E} = \frac{1}{c^2} \omega_{pe}^2 \underline{E} - \frac{1}{c^2} \omega^2 \underline{E}$$

$$-k^2 c^2 = \omega_{pe}^2 - \omega^2$$

$$\boxed{\omega^2 = \omega_{pe}^2 + k^2 c^2}$$

This is **electromagnetic dispersion relation**.



Remark

- If we let plasma density goes to zero, then we recover free space light wave

$$\omega = kc$$

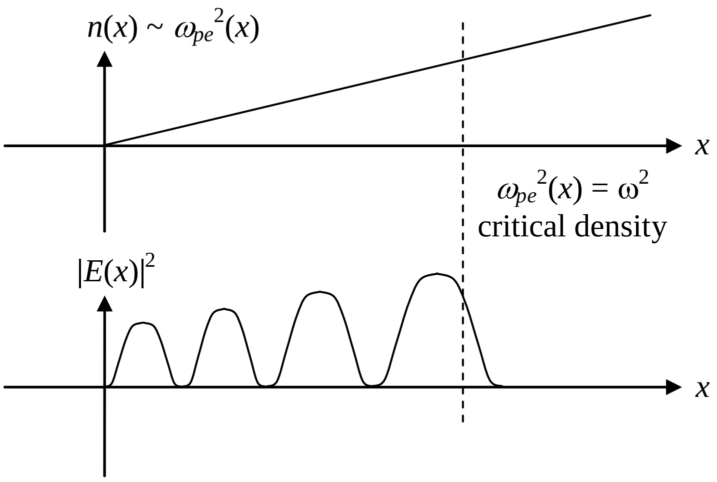
- Define **index of refraction**

$$n = \frac{kc}{\omega}$$

then for electromagnetic wave,

$$n = \frac{\sqrt{\omega^2 - \omega_{pe}^2}}{\omega} = \sqrt{1 - \omega_{pe}^2 / \omega^2}$$

As we can see, the index of refraction is imaginary when $\omega < \omega_{pe}$, which also corresponds to k being imaginary. What this mean is that wave with $\omega < \omega_{pe}$ cannot pass through plasma, and is reflected. The plasma density that makes $\omega_{pe} = \omega$ is called **critical density**. The total reflection of wave is termed **evanescence**. Unlike physical quantity, wave (or field) has to be continuous across boundary. Evanescent wave decays when it crosses the boundary to where the density is greater than the critical density.



- Application for this is when communication wave cannot penetrate through densely plasma, resulting in communication blackout while spacecraft is passing through ionosphere.

- Faraday's law suggests that

$$B \sim \frac{k}{\omega} E \sim \frac{1}{c} \frac{kc}{\omega} E \sim \frac{n}{c} E$$

Since for plasma, $n < 1$. Then $B << E$. Thus our assumption that we can set $\underline{v} \times \underline{B} = 0$ because magnetic force is much smaller than electric force is justified.

The first 3 waves: ion plasma wave or ion acoustic wave, electron plasma wave or Langmuir wave, and electromagnetic wave are the only waves for uniform unmagnetized plasma. Now we will look at waves for magnetized plasma.

Before we go further though, let's talk about some terminologies.

- Perpendicular wave: wave that $\underline{k} \perp \underline{B}_0$
- Parallel wave: wave that $\underline{k} \parallel \underline{B}_0$
- Transverse wave: wave that $\underline{k} \perp \underline{E}_1$
- Longitudinal wave: wave that $\underline{k} \parallel \underline{E}_1$
- Electrostatic wave: wave that $\underline{B}_1 = 0$
- Electromagnetic wave: wave that $\underline{B}_1 \neq 0$

Note that because of Faraday's law, $\underline{\nabla} \times \underline{E}_1 = -\partial_t \underline{B}_1$

- Longitudinal wave is electrostatic, and vice versa
- Transverse wave is electromagnetic, ***but not*** vice versa

4. Upper hybrid waves

This is

- high frequency wave (ion is in the background with density n_0)
- perpendicular longitudinal, hence electrostatic, wave ($\underline{k} \perp \underline{B}_0$, $\underline{k} \parallel \underline{E}_1$, and $\underline{B}_1 = 0$)

Equations that we need are

$$\partial_t n_e + \underline{\nabla} \bullet n_e \underline{v}_e = 0$$

$$m_e n_e \partial_t \underline{v}_e + m e_e n_e (\underline{v}_e \bullet \underline{\nabla}) \underline{v}_e = -e n_e \underline{E} - e n_e \underline{v}_e \times \underline{B}$$

$$\nabla \bullet E = \frac{e}{\varepsilon_0}(n_0 - n_e)$$

Let E_1 and k be in x-direction, and B_0 be in z-direction. Then linearization of the equations above give

$$-i\omega n_{e1} + ikn_0 v_x = 0 \qquad \text{(a)}$$

$$-i\omega m_e n_0 v_x = -e n_0 E_1 - e n_0 v_y B_0 \qquad \text{(b)}$$

$$-i\omega m_e n_0 v_y = e n_0 v_x B_0 \qquad \text{(c)}$$

$$ikE_1 = -\frac{e n_{e1}}{\varepsilon_0} \qquad \text{(d)}$$

Equation (d) gives

$$n_{e1} = -\frac{ik\varepsilon_0}{e} E_1.$$

Substituting this into (a) yields

$$v_x = -\frac{i\omega\varepsilon_0}{en_0} E_1.$$

Equation (c) gives

$$v_y = \frac{ieB_0}{\omega m_e} v_x$$

Substituting v_x into equation above gives

$$v_y = \frac{B_0\varepsilon_0}{m_e n_0} E_1$$

Substituting v_x and v_y into (b) yields

$$\omega^2 E_1 = \left(\frac{e^2 n_0}{\varepsilon_0 m_e}\right) E_1 + \left(\frac{e^2 B_0^2}{m_e^2}\right) E_1$$

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 \equiv \omega_{UH}^2}$$

This is the **dispersion relation for upper hybrid wave**.

Note that this frequency is independent of wave number k, and is reduced to cold electron plasma oscillation when B_0 approaches zero.

5. Electrostatic ion wave

This is

- low frequency wave (ion can no longer be ignored, i.e. $n_{e1} = n_{i1}$)
- longitudinal, hence electrostatic, wave ($\underline{k} \parallel \underline{E}_1$, and $\underline{B}_1 = 0$)

The equations we need are

$$m_e n_e \partial_t \underline{v}_e = -\gamma_e T_e \underline{\nabla} n_e - en_e \underline{E} - en_e \underline{v}_e \times \underline{B}$$

$$\partial_t n_e + \underline{\nabla} \bullet n_e \underline{v}_e = 0$$

$$m_i n_i \partial_t \underline{v}_i = -\gamma_i T_i \underline{\nabla} n_i + en_i \underline{E} + en_i \underline{v}_i \times \underline{B}$$

$$\partial_t n_i + \underline{\nabla} \bullet n_i \underline{v}_i = 0$$

v_e and v_i are by default first order quantities.

k can be in any direction of B_1, which we shall take it to be in z-direction. Let k be in the x-z plane so that $k_y = 0$.

Linearizing all 4 equations yield

$$-i\omega m_e n_0 \underline{v}_e = -i\underline{k}\gamma_e T_e n_{e1} - en_0 \underline{E}_1 - en_0 \underline{v}_e \times \underline{B}_0 \qquad \text{(a)}$$

$$-i\omega n_{e1} + i\underline{k} \bullet n_0 \underline{v}_e = 0 \qquad \text{(b)}$$

$$-i\omega m_i n_0 \underline{v}_i = -i\underline{k}\gamma_i T_i n_{i1} + en_0 \underline{E}_1 + en_0 \underline{v}_i \times \underline{B}_0 \qquad \text{(c)}$$

$$-i\omega n_{i1} + i\underline{k} \bullet n_0 \underline{v}_i = 0 \qquad \text{(d)}$$

Adding (a) and (c) gives

$$-i\omega n_0 \left(m_e \underline{v}_e + m_i v_i\right) = -i\underline{k}\left(\gamma_e T_e n_{e1} + \gamma_i T_i n_{i1}\right) + en_0 \left(\underline{v}_i - \underline{v}_e\right) \times \underline{B}_0 \qquad \text{(e)}$$

Applying $\underline{k} \bullet$ operation to (e)

$$-i\omega n_0 \underline{k} \bullet \left(m_e \underline{v}_e + m_i v_i\right) = -ik^2\left(\gamma_e T_e n_{e1} + \gamma_i T_i n_{i1}\right) + en_0 \underline{k} \bullet \left[\left(\underline{v}_i - \underline{v}_e\right) \times \underline{B}_0\right] \qquad \text{(f)}$$

Equations (b) and (d) say that $\underline{k} \bullet \underline{v}_e = \dfrac{\omega n_{e1}}{n_0}$ and $\underline{k} \bullet \underline{v}_i = \dfrac{\omega n_{i1}}{n_0} = \dfrac{\omega n_{e1}}{n_0}$, then (f) becomes

$$-i\omega^2 \left(m_e + m_i\right) n_{e1} = -ik^2 \left(\gamma_e T_e + \gamma_i T_i\right) n_{e1} + en_0 k_x \left(v_{iy} - v_{ey}\right) B_0 \qquad \text{(g)}$$

What we need now is to write v_{iy} and v_{ey} in term of n_{e1}. To do that, we apply $\underline{k} \times$ operation to (a) and get

$$-i\omega m_e n_0 \left(\underline{k} \times \underline{v}_e\right) = -en_0 \underline{k} \times \left(\underline{v}_e \times \underline{B}_0\right)$$

$$\underline{k} \times v_e = -\frac{i\omega_{ce}}{\omega} \underline{k} \times \left(\underline{v}_e \times \hat{z}\right)$$

which consists of 3 equations

$$-v_{ey} k_z = -\frac{i\omega_{ce}}{\omega} v_{ex} k_z$$

$$-v_{ez} k_x + v_{ex} k_z = -\frac{i\omega_{ce}}{\omega} v_{ey} k_z$$

$$v_{ey} k_x = \frac{i\omega_{ce}}{\omega} v_{ex} k_x$$

Solve for v_{ex} and v_{ez} in term of v_{ey}, then

$$v_{ex} = -\frac{i\omega}{\omega_{ce}} v_{ey}$$

$$v_{ez} = v_{ey} \frac{k_z}{k_x} i \frac{\omega_{ce}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right)$$

Also recall the continuity equation (b), we have

$$k_x v_{ex} + k_z v_{ez} = \omega \frac{n_{e1}}{n_0}.$$

Substituting v_{ex} and v_{ez} into the continuity equation, then

$$\left[-k_x \frac{i\omega}{\omega_{ce}} + \frac{k_z^2}{k_x} i \frac{\omega_{ce}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right)\right] v_{ey} = \omega \frac{n_{e1}}{n_0}$$

or

$$v_{ey} = \omega \frac{n_{e1}}{n_0} \left[-k_x \frac{i\omega}{\omega_{ce}} + \frac{k_z^2}{k_x} i \frac{\omega_{ce}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right)\right]^{-1}$$

Repeat the process for ion and get

$$v_{iy} = \omega \frac{n_{i1}}{n_0} \left[k_x \frac{i\omega}{\omega_{ci}} - \frac{k_z^2}{k_x} i \frac{\omega_{ci}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ci}^2}\right)\right]^{-1}$$

Substitute v_{ey} and v_{iy} into (g), rearranging the terms, and take $m_i + m_e \approx m_i$. Then

$$\boxed{1 - \frac{k^2 c_s^2}{\omega^2} + \frac{\omega_{ci}}{\omega} \left[\frac{1}{-\frac{\omega}{\omega_{ce}} - \frac{k_z^2}{k_x^2} \frac{\omega_{ce}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ce}^2}\right)} - \frac{1}{\frac{\omega}{\omega_{ci}} - \frac{k_z^2}{k_x^2} \frac{\omega_{ci}}{\omega} \left(1 - \frac{\omega^2}{\omega_{ci}^2}\right)} \right] = 0}$$

where $c_s^2 = (\gamma_e T_e + \gamma_i T_i)/m_i$. This is **dispersion relation for electrostatic ion wave**.

There are various limits that we can consider

- $\underline{k} \parallel \underline{B}_0$, then $k_x = 0$ and the dispersion relation is reduced to

$$1 - \frac{k^2 c_s^2}{\omega^2} = 0 \Leftrightarrow \omega^2 = k^2 c_s^2$$

This is exactly the ion acoustic wave, which makes sense because magnetic field does not affect parallel wave. Thus, we should recover the unmagnetized waves.

- $k_x >> k_z$, or wave propagates almost perpendicular to magnetic field. We can discard several terms with ω/ω_{ce} because we are only concern about low frequency wave. Thus, the dispersion relation becomes

$$1-\frac{k^2c_s^2}{\omega^2}+\frac{\omega_{ci}}{\omega}\left[\frac{1}{\frac{k_z^2}{k_x^2}\frac{\omega_{ce}}{\omega}}-\frac{1}{\frac{\omega}{\omega_{ci}}-\frac{k_z^2}{k_x^2}\frac{\omega_{ci}}{\omega}\left(1-\frac{\omega^2}{\omega_{ci}^2}\right)}\right]=0$$

$$1-\frac{k^2c_s^2}{\omega^2}+\frac{\omega_{ci}k_x^2}{\omega_{ce}k_z^2}-\frac{1}{\frac{\omega^2}{\omega_{ci}^2}-\frac{k_z^2}{k_x^2}\left(1-\frac{\omega^2}{\omega_{ci}^2}\right)}=0$$

If $\frac{k_x}{k_z} << \sqrt{\frac{m_i}{m_e}}$, then we can further simplify the dispersion relation

$$1-\frac{k^2c_s^2}{\omega^2}-\frac{\omega_{ci}^2}{\omega^2}=0$$

$$\boxed{\omega^2=k^2c_s^2+\omega_{ci}^2}$$

This is known as **electrostatic ion cyclotron wave dispersion relation**.

6. Lower hybrid waves

   Now let $\underline{k}$ be in x-direction, so $k_z=0$. Then

$$1-\frac{k^2c_s^2}{\omega^2}-\frac{\omega_{ci}\omega_{ce}}{\omega^2}-\frac{\omega_{ci}^2}{\omega^2}=0$$

   We can ignore the forth term because $|\omega_{ci}\omega_{ce}| >> \omega_{ci}^2$. The second term can be ignored for small k_x. Thus,

$$\boxed{1-\frac{\omega_{ci}\omega_{ce}}{\omega^2}=0 \Leftrightarrow \omega^2=\omega_{ci}\omega_{ce}\equiv\omega_{LH}^2}$$

   This is the **dispersion relation for lower hybrid waves**.

7. We have looked at electromagnetic wave in unmagnetized plasma. Now let's look at EM wave in magnetized plasma.

   This is a high frequency wave, so ions only appear in the background.

Equations which we need are, in linearized form,

$$-i\omega m_e n_0 \underline{v}_e = -en_0\underline{E}_1 - en_0\underline{v}_e \times \underline{B}_0$$

$$i\underline{k} \times \underline{E}_1 = i\omega \underline{B}_1$$

$$i\underline{k} \times \frac{\underline{B}_1}{\mu_0} = \underline{j}_1 - i\omega\varepsilon_0 \underline{E}_1$$

$$\underline{j}_1 = en_0\underline{v}_e$$

Let's look first for the perpendicular transverse wave where $\underline{k} \perp \underline{B}_0$ and $\underline{k} \perp \underline{E}_1$. There are 2 cases which are possible here:

- $\underline{E}_1 \parallel \underline{B}_0$: Ordinary wave

- $\underline{E}_1 \perp \underline{B}_0$: Extraordinary wave

In the first case, it can be seen that electrons will experience electric force and move in the same direction as B_0. Thus the term $\underline{v}_e \times \underline{B}_0$ will be zero or much less than the force due to electric field. This is exactly what we have for unmagnetized electromagnetic wave earlier. Thus the **dispersion relation for ordinary wave** is

$$\boxed{\omega^2 = \omega_{pe}^2 + k^2c^2}$$

as if there is no magnetic field present.

For the second case, the situation is rather complicate. I will not go into detail here, but the resulting **dispersion relation for extraordinary wave** is

$$\boxed{\left(1 - \frac{\omega^2}{\omega_{pe}^2}\right)\left(1 + \frac{k^2c^2}{\omega_{pe}^2} - \frac{\omega^2}{\omega_{pe}^2}\right) + \frac{\omega_{ce}^2 k^2 c^2}{\omega_{pe}^4} - \frac{\omega^2 \omega_{ce}^2}{\omega_{pe}^4} = 0}$$

or

$$\boxed{n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{UH}^2}}$$

There are 2 important properties which we should define for these O- and X-waves. They are

- **cutoff frequency** which corresponds to when *k* goes to zero.

- **resonance frequency** which corresponds to when k goes to $\pm\infty$

For resonance,

O-wave does not have resonance.

X-wave has resonance when $\omega = 0$ and when $\omega = \omega_{UH}$.

For cutoff,

O-wave has cutoff when $\omega = \omega_{pe}$

X-wave has cutoff when

$$1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{UH}^2} = 0$$

$$\omega^2\left(\omega^2 - \omega_{UH}^2\right) = \omega_{pe}^2\left(\omega^2 - \omega_{pe}^2\right)$$

$$\omega^4 - \omega_{UH}^2\omega^2 - \omega_{pe}^2\omega^2 + \omega_{pe}^4 = 0$$

$$\omega = \left[\frac{\omega_{UH}^2 + \omega_{pe}^2}{2} \pm \frac{1}{2}\sqrt{\left(\omega_{UH}^2 + \omega_{pe}^2\right)^2 - 4\omega_{pe}^4}\right]^{1/2}$$

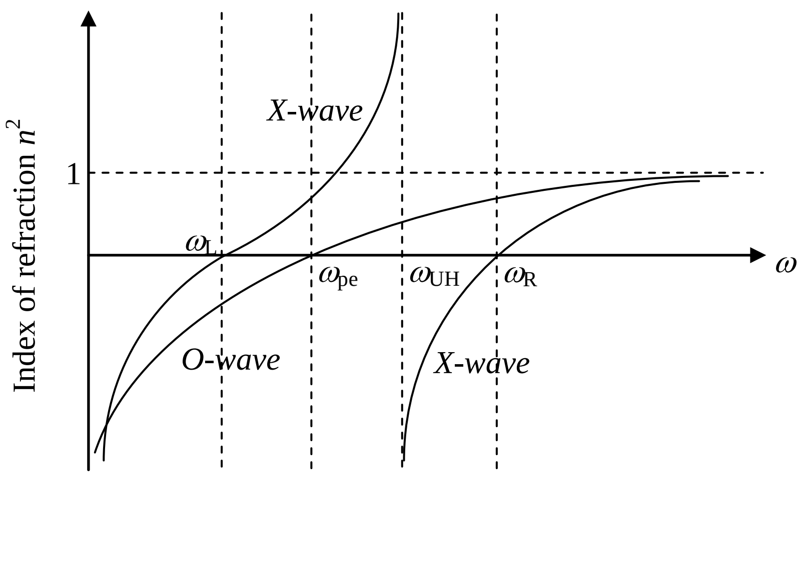
$$= \left[\omega_{pe}^2 + \frac{\omega_{ce}^2}{2} \pm \omega_{ce}\sqrt{\omega_{pe}^2 + \omega_{ce}^2/4}\right]^{1/2}$$

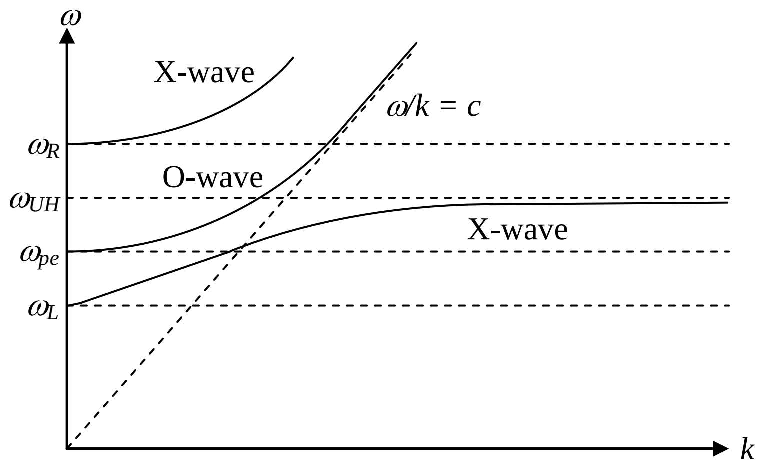
$$= \pm\frac{\omega_{ce}}{2} + \sqrt{\omega_{pe}^2 + \omega_{ce}^2/4}$$

We define

$$\omega_{\binom{L}{R}} = \pm\frac{\omega_{ce}}{2} + \sqrt{\omega_{pe}^2 + \omega_{ce}^2/4}$$

for **left and right hand cutoff**





ω
X-wave
ω/k = c
ω_R
O-wave
ω_UH
X-wave
ω_pe
ω_L
k