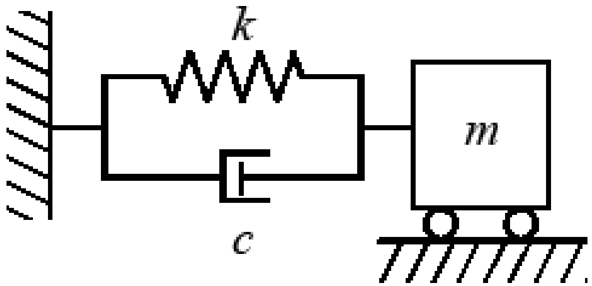
2.0 Outline

- Free Response of Undamped System
- Free Response of Damped System
- Natural Frequency, Damping Ratio

2.1 Free Response of Undamped System

Free vibration is the vibration of a system in response to initial excitations, consisting of initial displacements/ velocities. To obtain the free response, we must solve system of homogeneous ODEs, i.e. ones with zero applied forces. The standard form of MBK EOM is

$$m\ddot{x} + c\dot{x} + kx = 0, \quad x = x(t)$$



If the system is undamped, c = 0. The EOM becomes

$$m\ddot{x} + kx = 0 \Rightarrow \ddot{x} + \omega_n^2 x = 0, \quad \omega = \sqrt{k/m}$$

subject to the initial conditions $x(0) = x_0,\ \dot{x}(0) = v_0$

The solutions of homogeneous ODE are in the form

$x(t) = Ae^{st}$, A is the amplitude and s is constant

Subs. the solution into ODE, we get

$Ae^{st}\left(s^2 + \omega_n^2\right) = 0 \Rightarrow s^2 + \omega_n^2 = 0$ ** characteristic equation **

$s_{12} = \pm i\omega_n$ ** characteristic roots, eigenvalues **

general solution: $x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$ by superposition

We can now apply the given i.c. to solve for A_1 and A_2. However we will use some facts to arrange the solutions into a more appealing form.

Because $x(t)$ is real, $A_2 = \overline{A_1}$.

Let $A_1 = \frac{C}{2}e^{-i\phi}$. Therefore $A_2 = \frac{C}{2}e^{i\phi}$.

$$\therefore x(t) = \frac{C}{2}\left[e^{i(\omega_n t-\phi)} + e^{-i(\omega_n t-\phi)}\right] = C\cos(\omega_n t-\phi)$$

The given i.c. are then used to solve for the amplitude C and the phase angle Φ. Note ω_n, known as natural frequency, is the system parameter. The system is called *harmonic oscillator* because of its response to i.c. is the oscillation at harmonic frequency forever.

Ch. 2: Free Vibration of 1-DOF System

If the initial conditions are $x(0) = x_0$ and $\dot{x}(0) = v_0$

$$\therefore x_0 = C\cos\phi \text{ and } v_0 = \omega_n C\sin\phi \Rightarrow C = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \text{ and } \phi = \tan^{-1}\frac{v_0}{x_0\omega_n}$$

$\therefore x(t) = x_0 \cos\omega_n t + \dfrac{v_0}{\omega_n}\sin\omega_n t$ as the function of i.c. and system parameter

$T = 2\pi/\omega_n,\ f = 1/T$

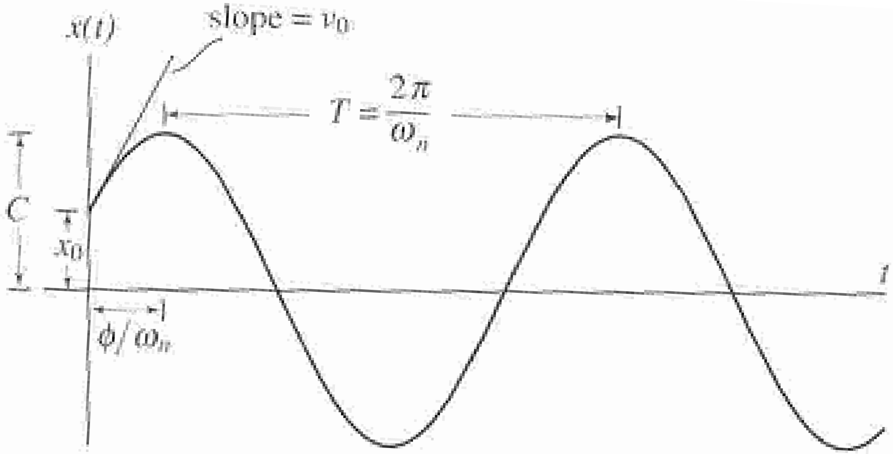


FIGURE 2.2
Response of a harmonic oscillator to initial excitations

2.1 Free Response of Undamped System

2.2 Free Response of Damped System

We normalize the standard MBK EOM by mass m:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\omega_n = \sqrt{k/m} =$ natural frequency

$\zeta = c/(2m\omega_n) =$ viscous damping factor

subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$

The solutions of homogeneous ODE are in the form

$x(t) = Ae^{st}$, A is the amplitude and s is constant

Subs. the solution into ODE, we get

$$Ae^{st}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)=0 \Rightarrow s^2+2\zeta\omega_n s+\omega_n^2=0$$ **CHE**

$$s_{12}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$ ** characteristic roots, eigenvalue **

general <u>solution</u>: $x(t)=A_1e^{s_1t}+A_2e^{s_2t}$ by superposition

Response of $m\ddot{x}+c\dot{x}+kx=0$, i.c. $x(0)=x_0$ and $\dot{x}(0)=v_0$

i) $\zeta=0$: $s_{12}=\pm i\omega_n$

$$x(t)=C\cos(\omega_n t-\phi)$$

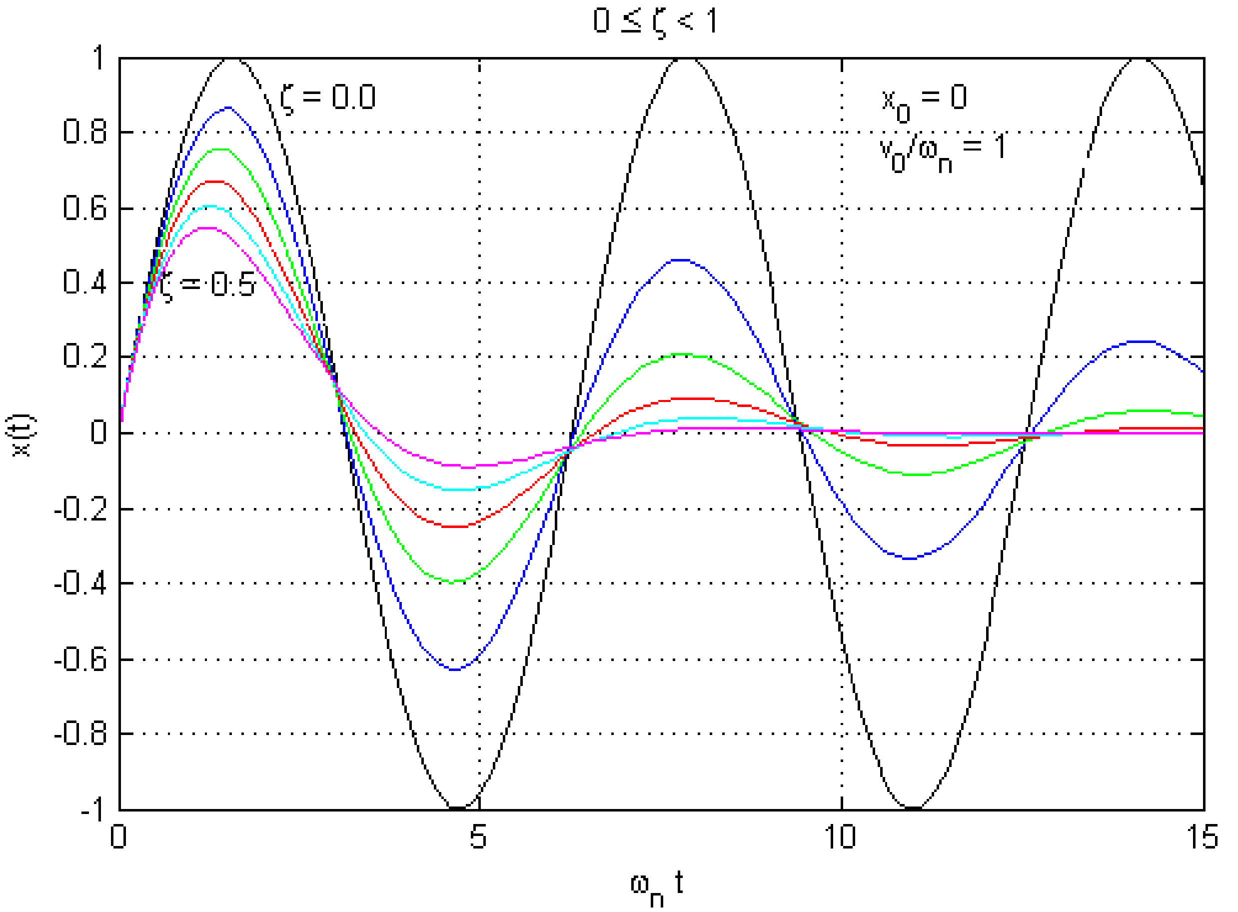
harmonic oscillation with frequency ω_n

ii) $0<\zeta<1$: $s_{12}=-\zeta\omega_n \pm i\omega_d$ where $\omega_d=\omega_n\sqrt{1-\zeta^2}$

$$x(t)=Ce^{-\zeta\omega_n t}\cos(\omega_d t-\phi)$$

exponentially decaying amplitude oscillation

with damped frequency ω_d and envelope $Ce^{-\zeta\omega_n t}$



2.2 Free Response of Damped System

Response of $m\ddot{x}+c\dot{x}+kx=0$, i.c. $x(0)=x_0$ and $\dot{x}(0)=v_0$

iii) $\zeta>1$: $s_{12}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$

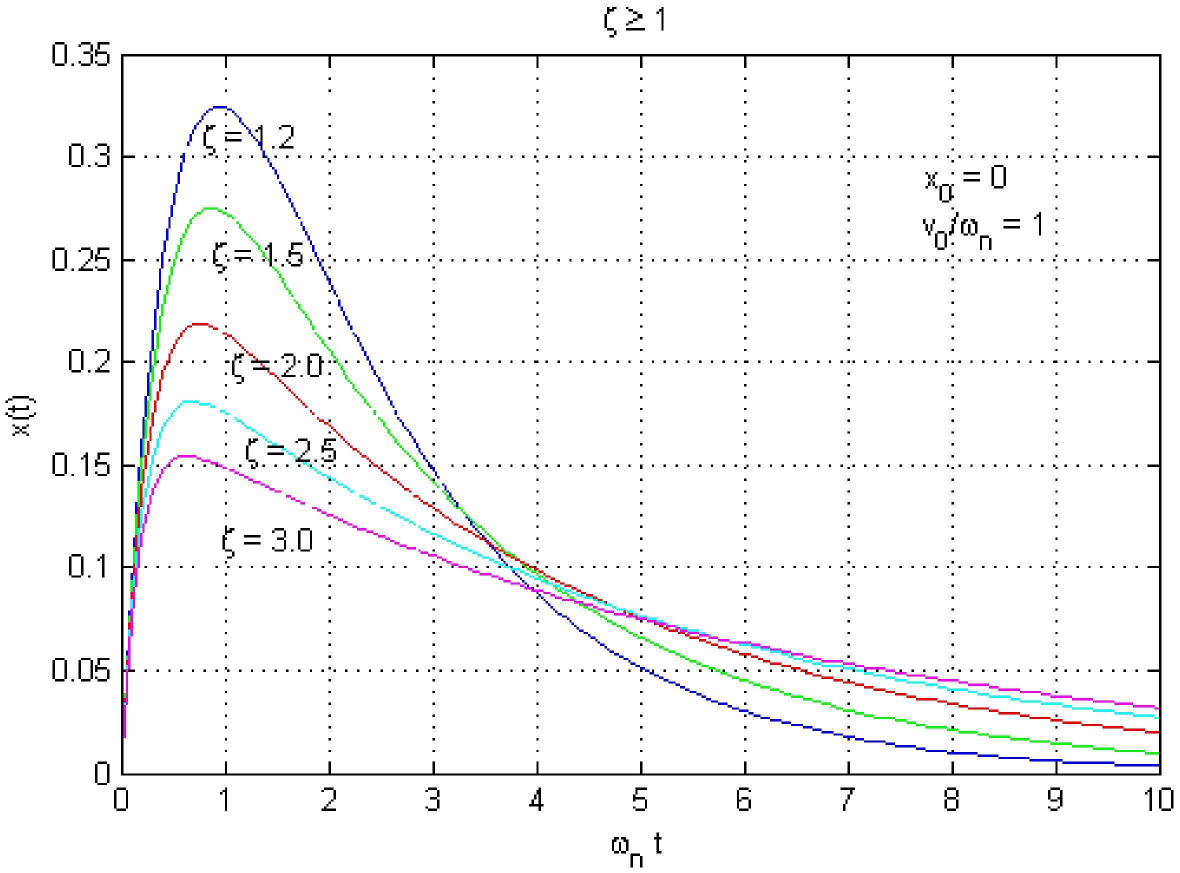
$$x(t)=e^{-\zeta\omega_n t}\left(A_1e^{-\omega_n t\sqrt{\zeta^2-1}}+A_2e^{\omega_n t\sqrt{\zeta^2-1}}\right)$$

aperiodic decay with peak more suppressed and decay further slow down as ζ increase

iv) $\zeta=1$: $s_{12}=-\omega_n$

$$x(t)=(A_1+A_2t)e^{-\omega_n t}$$

aperiodic decay with highest peak and fastest decay


ζ ≥ 1
ζ = 1.2
ζ = 1.5
ζ = 2.0
ζ = 2.5
ζ = 3.0
x0 = 0
v0/ωn = 1
x(t)
ωn t
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10

2.3 Natural Frequency, Damping Ratio

The response will partly be dictated by the roots s_{12}, which depend on ζ and ω_n. Root locus diagram gives a complete picture of the manner in which s_{12} change with ζ and ω_n. The focus will be the left half s-plane where the system response is stable.

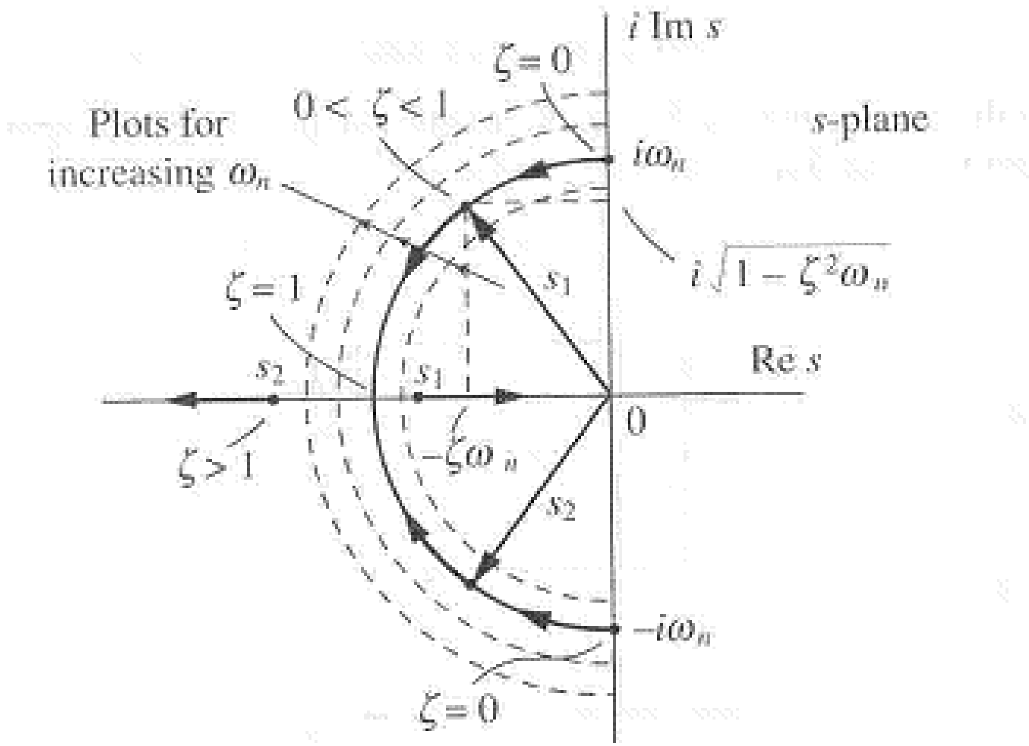


FIGURE 2.7
Root-locus diagram for a viscously damped system

2.3 Natural Frequency, Damping Ratio

ω_n constant, vary ζ

i) $\zeta = 0$ (undamped), $s_{12} = \pm i\omega_n$ on the Im-axis far from origin by ω_n. The motion is *harmonic oscillation* with natural frequency ω_n.

ii) $0 < \zeta < 1$ (underdamped), $s_{12} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$ pair of symmetric points moving on a semicircle of radius ω_n. The motion is *oscillatory decay*.

iii) $\zeta = 1$ (critically damped), $s_{12} = -\omega_n$ repeated roots. The motion is *aperiodic decay*.

iv) $\zeta > 1$ (overdamped), $s_{12} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

two negative real roots going to 0 and $-\infty$.

The motion is *aperiodic decay*.

ζ constant, vary ω_n

The symmetric roots will be far from origin

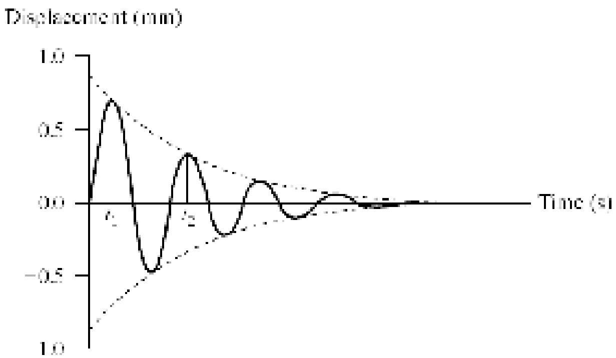
along the radius ω_n making an angle $\cos^{-1}(\zeta)$ with -*x* axis.

Determination of M-B-K

- Mass – directly measure the weight or deduced from frequency of oscillation
- Spring – from measures of the force and deflection or deduced from frequency of oscillation
- Damper – deduced from the decrementing response

m and k can be determined easily by static tests.

c ? ⇨ Determined by the concept of logarithmic decrement

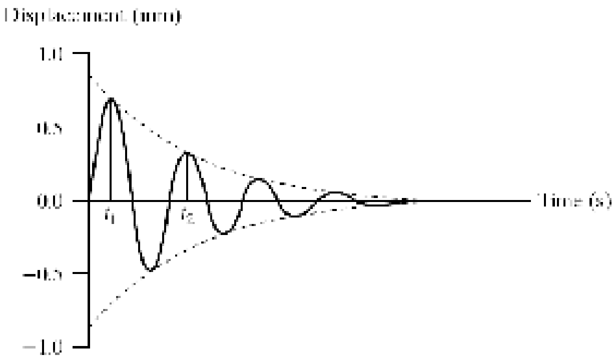


Define logarithmic decrement δ

$$\delta = \ln \frac{x(t)}{x(t+T)}$$

Underdamped motion ⇨

$$\delta = \ln \frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n (t+T)}\sin(\omega_d t + \omega_c T + \phi)}$$



$$\delta = \ln \frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n (t+T)}\sin(\omega_d t + \omega_c T + \phi)}$$

$$\delta = \ln e^{\zeta\omega_n T} = \zeta\omega_n T$$

$$T = \frac{2\pi}{\omega_d} \quad \Rightarrow \quad \delta = \zeta\omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Measurement done over any integer multiple of the period

$$\delta = \frac{1}{n}\ln\frac{x(t)}{x(t+nT)}$$

Ex. 1 A given system of unknown mass m and spring k was observed to oscillate harmonically in free vibration with $T_n = 2\pi \times 10^{-2}$ s. When a mass M = 0.9 kg was added to the system, the new period rose to $2.5\pi \times 10^{-2}$ s. Determine the system parameters m and k.

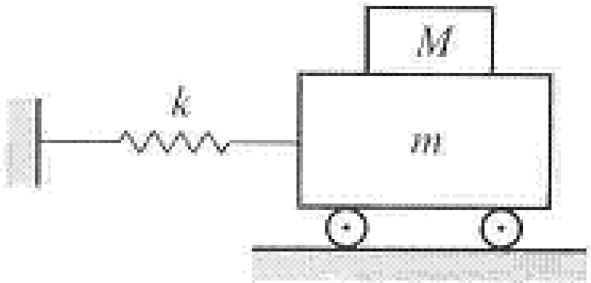
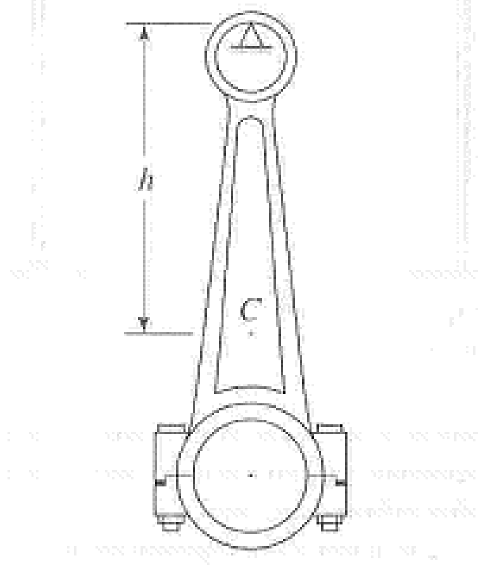


FIGURE 2.17
System with unknown mass m and spring k

Ex. 2 A connecting rod of mass $m = 3 \times 10^{-3}$ kg and $I_C = 0.432 \times 10^{-4}$ kgm^2 is suspended on a knife edge about the upper inner surface of a wrist-pin bearing, as shown in the figure. When disturbed slightly, the rod was observed to oscillate harmonically with $\omega_n = 6$ rad/s. Determine the distance h between the support and the C.M.



$m = 3\times10^{-3}$ kg, $I_C = 0.432\times10^{-4}$ kgm^2, $\omega_n = 6$ rad/s

$$\omega_n^2 = \frac{mgh}{I_C + mh^2} \Rightarrow h = 0.2,\ 0.072 \text{ m}$$

Radius of gyration $= \sqrt{\dfrac{I_C}{m}} = 0.12$ m must be greater than the longest length of the object. $\therefore h = 0.072$ m

Ex. 3 A disk of mass m and radius R rolls w/o slip while restrained by a dashpot with coefficient of viscous damping c in parallel with a spring of stiffness k. Derive the differential equation for the displacement x(t) of the disk mass center C and determine the viscous damping factor ζ and the frequency ω_n of undamped oscillation.

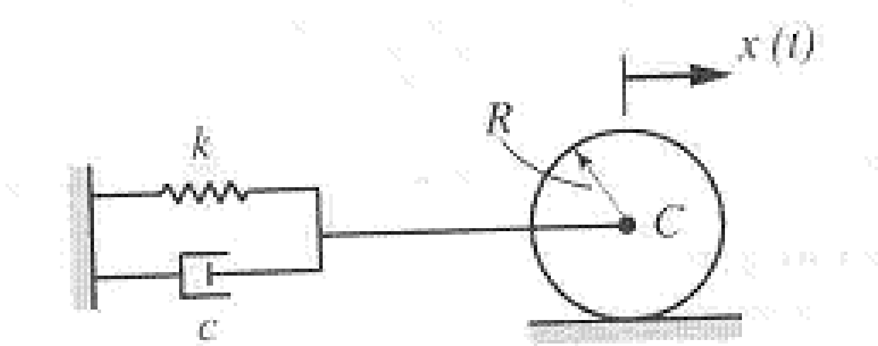
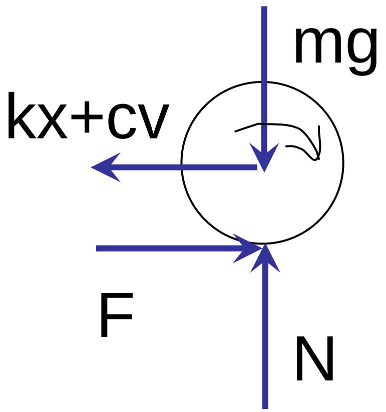


FIGURE 2.22
Rolling disk restrained by a spring and a dashpot



$$\left[\sum M_C = I_C\ddot{\theta}\right] \quad -FR = I_C\frac{\ddot{x}}{R}$$

$$\left[\sum F_x = m\ddot{x}\right] \quad F - kx - c\dot{x} = m\ddot{x}$$

$$\left(m + \frac{I_C}{R^2}\right)\ddot{x} + c\dot{x} + kx = 0, \; I_C = m\frac{R^2}{2}$$

$$\zeta = \frac{c}{\sqrt{6km}}, \; \omega_n = \sqrt{\frac{2k}{3m}}$$

2.3 Natural Frequency, Damping Ratio

Ex. 4 Calculate the frequency of damped oscillation of the system for the values m = 1750 kg, c = 3500 Ns/m, k = 7x10^5 N/m, a = 1.25 m, and b = 2.5 m. Determine the value of critical damping.

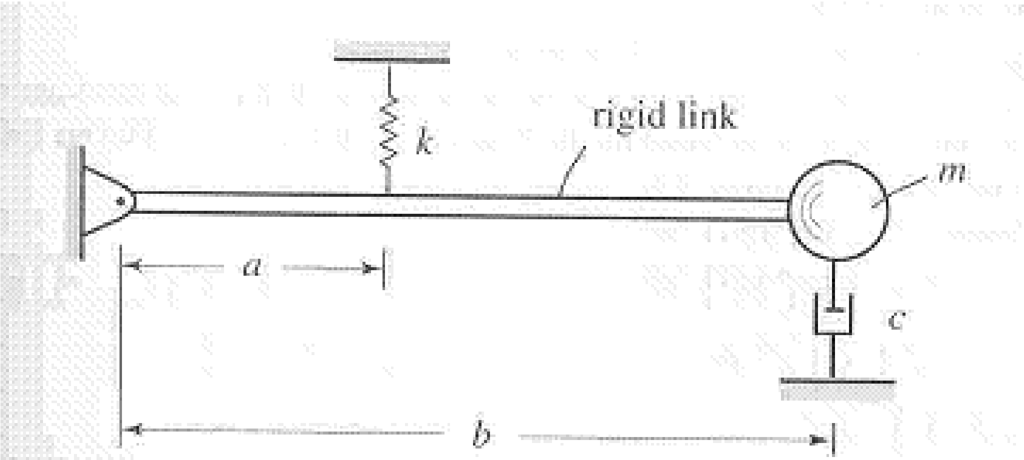


FIGURE 2.23

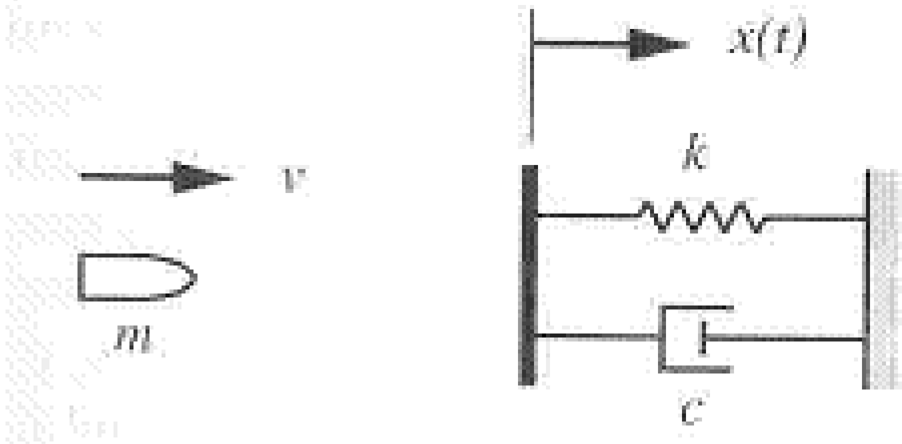
$$m\ddot{y} + c\dot{y} + \frac{a^2}{b^2}ky = 0 \quad (\text{weaken spring})$$

$$\omega_n = \frac{a}{b}\sqrt{\frac{k}{m}} = 10 \text{ rad/s} \quad \zeta = \frac{c}{2m\omega_n} = 0.1$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 9.95 \text{ rad/s}$$

$$c_{cr} = 2m\omega_n = 35,000 \text{ Ns/m}$$

Ex. 5 A projectile of mass m = 10 kg traveling with v = 50 m/s strikes and becomes embedded in a massless board supported by a spring stiffness k = 6.4x10^4 N/m in parallel with a dashpot of c = 400 Ns/m. Determine the time required for the board to reach the max displacement and the value of max displacement.



$$\omega_n = \sqrt{\frac{k}{m}} = 80 \text{ rad/s} \quad \zeta = \frac{c}{2m\omega_n} = 0.25 < 1 \;\therefore \text{underdamped}$$

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$$\dot{x}(t) = -\zeta\omega_n Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi) - \omega_d Ce^{-\zeta\omega_n t}\sin(\omega_d t - \phi)$$

$$x(0) = C\cos\phi = 0, \;\; \phi = \pi / 2$$

$$\dot{x}(0) = 50 = C\omega_n\sqrt{1-\zeta^2}, \;\; C = 0.6455$$

max displacement when $\dot{x}(t) = 0 \Rightarrow \tan\omega_d t = \dfrac{\omega_d}{\zeta\omega_n}, \;\; t = 0.017 \text{ s}$

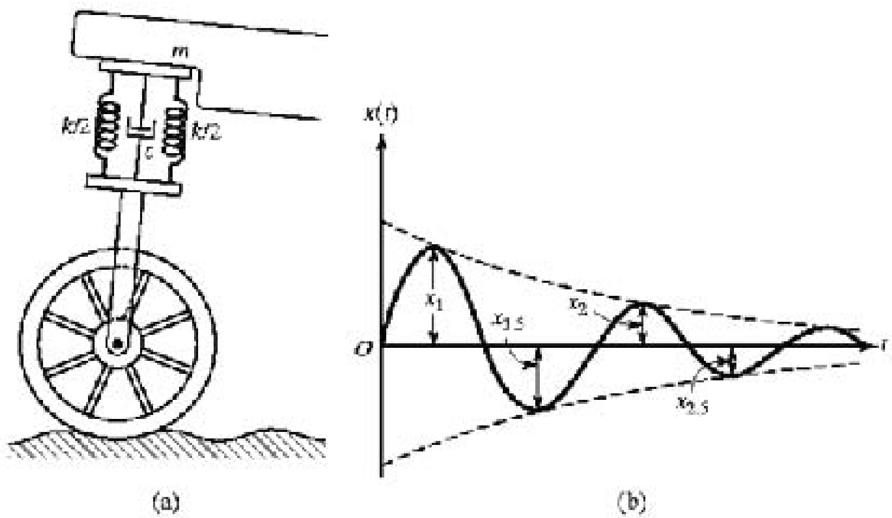
$$x(0.017) = 0.4447 \text{ m}$$

Example 2.2 (from Rao's)
Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200kg (shown in Fig.(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig.(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Example 2.2 Solution

Approach: We use the equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.



2.3 Natural Frequency, Damping Ratio

Example 2.2 Solution

Since $x_{1.5} = x_1 / 4, \qquad x_2 = x_{1.5} / 4 = x_1 / 16$

Hence the logarithmic decrement becomes

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

From which ζ can be found as 0.4037. The damped period of vibration given by 2 s. Hence,

$$2 = \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1-(0.4037)^2}} = 3.4338 \quad \text{rad/s}$$

2.3 Natural Frequency, Damping Ratio

Example 2.2 Solution

The damping constant can be obtained:

$$c = 2\zeta\omega_n m = 2(0.4037)(3.4338)(200) = 554.4981\ \text{N-s/m}$$

and the stiffness by:

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652\ \text{N/m}$$

The displacement of the mass will attain its max value at time t_1, given by

$$\sin \omega_d t_1 = \sqrt{1-\zeta^2}$$

This gives:

$$\sin \omega_d t_1 = \sin \pi t_1 = \sqrt{1-(0.4037)^2} = 0.9149$$

or

$$t_1 = \frac{\sin^{-1}(0.9149)}{\pi} = 0.3678\ \text{sec}$$

Example 2.2 Solution

The envelope passing through the max points is:

$$x = \sqrt{1-\zeta^2}\,Ae^{-\zeta\omega_n t}$$

Since x = 250mm,

$$0.25 = \sqrt{1-(0.4037)^2}\,Ae^{-(0.4037)(3.4338)(0.3678)}$$

$$A = 0.4550 \text{ m}$$

The velocity of mass can be obtained by differentiating the displacement:

$$x(t) = Ae^{-\zeta\omega_n t}\sin\omega_d t$$

$$\dot{x}(t) = Ae^{-\zeta\omega_n t}(-\zeta\omega_n \sin\omega_d t + \omega_d\cos\omega_d t)$$

as

When t = 0,

$$\dot{x}(t=0) = \dot{x}_0 = A\omega_d = A\omega_n\sqrt{1-\zeta^2} = (0.4550)(3.4338)\sqrt{1-(0.4037)^2}$$

$$= 1.4294 \text{ m/s}$$