Multiple Equations Model

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Covered Topics (1)

Model Components

- Simultaneous Equations
  - –Endogenous variables
  - –Exogenous variables
- Cross-equation Correlation
- Cross-equation Restriction

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Covered Topics (2)

Estimation Methods for MEM

- Equation-by-equation estimation
- SURE
- Three-stage LS
- GMM

Test for Exogeneity ???

Simultaneous Equations Model (1)

Or **SEM** for short

Exogenous Variable

- its value is given

Endogenous Variable

- its value determined by the other variables (exog. or other endo.)

Simultaneous Equations Model (2)

For an *M* equations model,

there can only be *M* endogenous variables. The other must be exogenous.

Random components or error terms must be exogenous, of course.

Simultaneous Equations Model (3)

Q: Who will say which variables in the model are endogenous and which are exogenous?

A: Generally, their ex/endogeneity is determined by the model structure. Easily speaking, they are assumed. However, they could be tested for exogeneity.

Structural Equations

For a non-structural equation, variables can be on any side of the equations. But for a structural equation, the variable on the left-hand side is generally endogenous. Its value will be determined by the variables on the right-hand side.

Model with *M* Independent Structural Equations (1)

$$Y_{1i} = \beta_{11} X_{1i} + \beta_{12} X_{2i} + \cdots + \beta_{1K} X_{Ki} + \varepsilon_{1i} \quad \text{- -(1)}$$

$$\vdots$$

$$Y_{Mi} = \beta_{M1} X_{1i} + \beta_{M2} X_{2i} + \cdots + \beta_{MK} X_{Ki} + \varepsilon_{Mi} \quad \text{- -(M)}$$

Y - endogenously determined by X and ε

X - exogenously given

ε - randomly determined with zero mean.

Model with *M* Independent Structural Equations (2)

ε - uncorrelated between equations

$$\mathrm{E}(\varepsilon) = 0$$

$$\mathrm{cov}(\varepsilon_{mi}, \varepsilon_{nj}) = 0$$

$$\text{for all} \quad m, n = 1, \ldots, M$$

$$\text{and all} \quad i, j$$

==> equation-by-equation estimation (unrelated regression) is valid and BLUE

SUR Model (1)

Seeming **U**nrelated **R**egression Model

$$Y_{1i} = \beta_{11} X_{1i} + \beta_{12} X_{2i} + \cdots + \beta_{1K} X_{Ki} + \varepsilon_{1i} \quad \text{--(1)}$$

$$\vdots$$

$$Y_{Mi} = \beta_{M1} X_{1i} + \beta_{M2} X_{2i} + \cdots + \beta_{MK} X_{Ki} + \varepsilon_{Mi} \quad \text{--(M)}$$

$$\mathrm{cov}(\varepsilon_{mi}, \varepsilon_{ni}) \neq 0 \quad \text{for some} \quad m \neq n$$

$$\text{but } \mathrm{cov}(\varepsilon_{mi}, \varepsilon_{nj}) = 0 \quad \text{for all} \quad i \neq j$$

SUR Model (2)

SUR Model in Matrix Form

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{Y} = \begin{bmatrix} Y_{11} & \cdots & Y_{M1} \\ \vdots & \vdots & \vdots \\ Y_{1N} & \cdots & Y_{MN} \end{bmatrix}_{\text{NxM}}$$

$$= \begin{bmatrix} \mathbf{Y}_{1.} & \cdots & \mathbf{Y}_{M.} \end{bmatrix}$$

SUR Model (3)

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdots & X_{K1} \\ \vdots & \vdots & \vdots \\ X_{1N} & \cdots & X_{KN} \end{bmatrix}_{\text{NxK}},$$

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{M1} \\ \vdots & \vdots & \vdots \\ \beta_{1K} & \cdots & \beta_{MK} \end{bmatrix}_{\text{KxM}} \rightarrow \boldsymbol{\beta}_{\mathbf{M.}}$$

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SUR Model (4)

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{21} & \cdots & \varepsilon_{M1} \\ \varepsilon_{12} & \varepsilon_{22} & \cdots & \varepsilon_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{1N} & \varepsilon_{2N} & \cdots & \varepsilon_{MN} \end{bmatrix}_{\mathrm{NxM}} = \begin{bmatrix} \boldsymbol{\varepsilon}_{.1} \\ \boldsymbol{\varepsilon}_{.2} \\ \vdots \\ \boldsymbol{\varepsilon}_{.N} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{\varepsilon}_{1.} & \boldsymbol{\varepsilon}_{2.} & \cdots & \boldsymbol{\varepsilon}_{M.} \end{bmatrix}$$

SUR Model (5)

$$Var(\boldsymbol{\varepsilon}_{\mathbf{i}.}) = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \cdots & \sigma_{M1} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1M} & \sigma_{2M} & \cdots & \sigma_{MM} \end{bmatrix}_{\mathrm{MXM}}$$

$$\text{for all i} = 1, \cdots, N$$

$$\Sigma$$

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$X\beta$

SUR Model (6)

Stacked form

$$\underbrace{\begin{bmatrix} Y_{1.} \\ Y_{2.} \\ \vdots \\ Y_{M.} \end{bmatrix}}_{Y} = \begin{bmatrix} X\beta_{1.} & 0 & 0 & 0 \\ 0 & X\beta_{2.} & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & X\beta_{M.} \end{bmatrix} + \underbrace{\begin{bmatrix} \varepsilon_{1.} \\ \varepsilon_{2.} \\ \\ \varepsilon_{M.} \end{bmatrix}}_{e}$$

SUR Model (7)

$$Var(\boldsymbol{e}) = \begin{bmatrix} \begin{matrix} \sigma_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{11} \end{matrix} & \begin{matrix} \sigma_{21} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{21} \end{matrix} & \cdots & \begin{matrix} \sigma_{M1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{M1} \end{matrix} \\ \begin{matrix} \sigma_{12} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{12} \end{matrix} & \begin{matrix} \sigma_{22} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{22} \end{matrix} & \cdots & \begin{matrix} \sigma_{M2} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{M2} \end{matrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} \sigma_{1M} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{1M} \end{matrix} & \begin{matrix} \sigma_{2M} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{2M} \end{matrix} & \cdots & \begin{matrix} \sigma_{MM} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{MM} \end{matrix} \end{bmatrix}_{MNXMN}$$

$$= \boldsymbol{\Sigma} \otimes \mathbf{I}_N \qquad \text{(Kronecker Product)}$$

SUR Model (8)

<u>Estimation Method</u>

Step 0 Set $\boldsymbol{\Sigma}$=$\mathbf{I}$ (MxM) => $\boldsymbol{\Omega}$=$\mathbf{I}$ (MNxMN)

Step 1 Run GLS on the "giant" stacked problem

$$\hat{\beta} = [X^T(\hat{\Sigma}^{-1} \otimes I_N)X]^{-1} X^T(\hat{\Sigma}^{-1} \otimes I_N)Y \quad \text{(MKx1vector)}$$

Step 2 Estimate Σ matrix

$$\hat{\sigma}_{mn} = \frac{\left[\mathbf{Y}_{m.} - \mathbf{X}\hat{\beta}_{m.}\right]^T \left[\mathbf{Y}_{n.} - \mathbf{X}\hat{\beta}_{n.}\right]}{N}, \; m, n = 1, \ldots, M$$

SUR Model (9)

<u>Estimation Method</u> (cont'd)

Step 3 Recalculate Ω

$$\boldsymbol{\Omega}^T\boldsymbol{\Omega} = \left[\hat{\boldsymbol{\Sigma}} \otimes \mathbf{I}_N\right]^{-1}$$

$$= \hat{\boldsymbol{\Sigma}}^{-1} \otimes \mathbf{I}_N$$

$$\boldsymbol{\Omega} = \hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}} \otimes \mathbf{I}_N$$

Go back to step 1 until convergence is achieved.

Note that SUR is also applied to Panel Data Analysis.

SUR with Cross-equation Restriction

Linear Restriction **Rβ=r**

In Step 1 above, starting from the stacked form **Y=Xβ**+*e*, apply weight matrix **Ω**

==>ΩY=ΩXβ+Ω*e*

Apply RLS to the weighted stacked form. See "Inference" module for RLS.

SEM with Cross-equation Correlation (1)

$$Y_{1i} = \gamma_{12}Y_{2i} + \cdots + \gamma_{1M}Y_{Mi} + \beta_{11}X_{1i} + \beta_{12}X_{2i} + \cdots + \beta_{1K}X_{Ki} + \varepsilon_{1i}$$

$$\vdots$$

$$Y_{Mi} = \gamma_{M1}Y_{1i} + \cdots + \gamma_{M,M-1}Y_{M-1,i} + \beta_{M1}X_{1i} + \beta_{M2}X_{2i} + \cdots + \beta_{MK}X_{Ki} + \varepsilon_{Mi}$$

$\boxed{\gamma \neq 0}$

Y - endogenously determined by X and ε

X - exogenously given

SEM with Cross-equation Correlation (2)

ε - randomly determined with zero mean.

ε - uncorrelated between equations

$$\mathrm{E}(\varepsilon) = 0$$

$$\operatorname{cov}(\varepsilon_{mi}, \varepsilon_{nj}) = 0$$

$$\text{for all} \quad m, n = 1, \dots, M$$

$$\text{and all} \quad i, j$$

SEM with Cross-equation Correlation (3)

$$\mathbf{V}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1\mathrm{M}} \\ \sigma_{12} & \sigma_{22} & \ddots & \sigma_{2\mathrm{M}} \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{1\mathrm{M}} & \sigma_{2\mathrm{M}} & \cdots & \sigma_{\mathrm{MM}} \end{bmatrix}$$

=> equation-by-equation estimation is **invalid**.
Why?

SEM with Cross-equation Correlation (4)

Assume that X is non-random or

$$\text{cov}(X_{ki}, \varepsilon_{mi}) = 0 \quad \text{for all} \quad m = 1,\dots, M$$
$$k = 1,\dots, K \text{ and all} \quad i = 1,\dots, N$$

SEM with Cross-equation Correlation (5)

$$\text{cov}(Y_{mi}, \varepsilon_{ni}) = \text{cov}(\cdots + \gamma_{mn} Y_{ni} + \cdots + \varepsilon_{mi}, \varepsilon_{ni})$$
$$= \cdots + \gamma_{mn} \text{cov}(Y_{ni}, \varepsilon_{ni}) + \cdots + \text{cov}(\varepsilon_{mi}, \varepsilon_{ni})$$
$$= \cdots + \gamma_{mn} \text{cov}(\cdots\cdots + \varepsilon_{ni}, \varepsilon_{ni}) + \cdots + \text{cov}(\varepsilon_{mi}, \varepsilon_{ni})$$
$$= \cdots + \gamma_{mn} \text{var}(\varepsilon_{ni}) + \cdots + \text{cov}(\varepsilon_{mi}, \varepsilon_{ni})$$

That is, $\text{cov}(Y_{mi}, \varepsilon_{ni}) \neq 0$, in general,

for all $m \neq n$.

==> violation of CLRM assumption

SEM with Cross-equation Correlation (6)

Valid Estimation Methods

- equation-by-equation 2SLS. Require a set of instrument variables(IV's) which include all the X's plus outside variables (if needed). Endogenous variables (Y's) cannot be IV.
- Simultaneous Approaches. Estimate all the equations simultaneously.

SEM in Matrix Form

$$\mathbf{Y} = \mathbf{Y\Gamma} + \mathbf{X\beta} + \boldsymbol{\varepsilon}$$

$$\text{where}\quad \mathbf{\Gamma} = \begin{bmatrix} 0 & \gamma_{21} & \cdots & \gamma_{M1} \\ \gamma_{12} & 0 & \cdots & \gamma_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1M} & \gamma_{2M} & \cdots & 0 \end{bmatrix}$$

Identification Problems (1)

No matter which approach is to be used (eq-by-eq or simultaneous), this question need to be answered first.

Can all the unknown parameters (γ,β) in each equation be validly estimated?

What could be the problems?

Identification Problems (2)

The number of parameters to be estimated could exceed the number of available IV or exogenous variables.

Check Identification problems

- Exact Identification ==> proceed
- Over Identification ==> proceed
- Under Identification ==>need extra IV's

Identification Problems (3)

Identification Problem need to be checked equation by equation. That is, one equation could be exactly identified while the other is under-identified.

Identification Problems (4)

M = Max. number of endogenous variables in any equation (it is the number of eq's)

K = Max. number of exogenous variables in any equation

m_m = number of γ parameters in equation *m* needed to be estimated (max. is M-1)

k_m = number of β parameters in equation *m* needed to be estimated (max. is K)

Identification Problems (5)

<u>Order Conditions</u>

Equation *m* is

- exactly identified if $K = m_m + k_m$
- over-identified if $K > m_m + k_m$
- under identified if $K < m_m + k_m$

That is, check for # of (γ, β) parameters to be estimated in each equation against K.

Identification Problems (6)

<u>Rank Conditions</u>

Define U_m = sub-matrix of $\begin{bmatrix} \mathbf{I} - \mathbf{\Gamma} \\ -\boldsymbol{\beta} \end{bmatrix}$ which includes only the rows corresponding to Y and X not in equation *m*.

Equation *m* is

- identified if $\text{Rank}(U_m) = M-1$
- unidentified if $\text{Rank}(U_m) < M-1$

Identification Problems (7)

Equation-by-equation approach

order condition is enough

Simultaneous approach

order condition is necessary

rank condition is sufficient

Identification Problems (8)

ID Problem with intra-equation parameter restriction

r_m = number of non-redundant restriction on (γ,β) parameters in equation *m*, e.g.,

$$\gamma_{m2} + \beta_{m3} = 1$$

$$\beta_{m2} = \beta_{m4}$$

Check $(m_m + k_m - r_m)$ against K.

Cross-eq restriction =>eqxeq method fails

2SLS (1)

The Same Two-Stage Least Square as in IV section. It is used eqn-by-eqn estn.

<u>Assumption</u>

1) all the X's are uncorrelated with all the ε's.

2) there could be extra IV's from outside model.

2SLS (2)

For simplicity's sake, all IV could be assumed to be non-correlated with the error vector (all error terms in all the equation). More complex if some variables are correlated with some error terms but not with the others, e.g.,

$$Cov(X_{2i}, \varepsilon_{3i}) = 0 \quad \text{but} \quad Cov(X_{2i}, \varepsilon_{4i}) \neq 0$$

2SLS (3)

Define

$\mathbf{Z}_m$ = matrix of IV's for equation *m*

That is, $\mathrm{cov}(\mathbf{Z}_m, \varepsilon_m)=0$

Stage 1 For each equation *m*, regress $\mathbf{Y}_m$ on $\mathbf{Z}_m$ using OLS. Get fitted value of Y_m.

2SLS (4)

$$\mathbf{A}_m = \mathbf{Z}_m[\mathbf{Z}_m^T\mathbf{Z}_m]^{-1}\mathbf{Z}_m^T, \mathbf{A}_m = \mathbf{A}_m^T = \mathbf{A}_m^2$$

$$\hat{\mathbf{Y}}_{m.} = \mathbf{A}_m\mathbf{Y}_{m.}$$

Stage 2 Substitute $Y_{m.}$ with $\hat{\mathbf{Y}}_{m.}$. Then, estimate $\gamma_{m.}$, $\beta_{m.}$ with OLS

Estimate

$$\hat{\mathbf{\Sigma}} = \frac{\begin{bmatrix} \mathbf{I} - \hat{\mathbf{\Gamma}}^T & -\hat{\boldsymbol{\beta}}^T \end{bmatrix} \begin{bmatrix} \mathbf{Y}^T \\ \mathbf{X}^T \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{X} \end{bmatrix} \begin{bmatrix} \mathbf{I} - \hat{\mathbf{\Gamma}} \\ -\hat{\boldsymbol{\beta}} \end{bmatrix}}{\mathrm{N}}$$

Indirect Least Square (1)

Or **ILS** for short.

$$\mathbf{Y}[\mathbf{I}-\mathbf{\Gamma}]=\mathbf{X}\beta+\boldsymbol{\varepsilon}$$

$$\mathbf{Y}=\mathbf{X}\beta[\mathbf{I}-\mathbf{\Gamma}]^{-1}+\boldsymbol{\varepsilon}[\mathbf{I}-\mathbf{\Gamma}]^{-1}$$

$$=\mathbf{X}\Pi+\xi$$ Reduced Form

where $\Pi=\beta[\mathbf{I}-\mathbf{\Gamma}]^{-1}$ KxM matrix

$$\xi=\boldsymbol{\varepsilon}[\mathbf{I}-\mathbf{\Gamma}]^{-1}$$

Indirect Least Square (2)

Note that

1) Var(ξ) is not diagonal but Var(**ε**) is.

2) $\mathbf{Var}(\xi)=[\mathrm{I}-\Gamma^{\mathrm{T}}]^{-1}\Sigma\ [\mathrm{I}-\Gamma]^{-1}$

3) if Σ and Γ are known, use SUR with

$$\mathbf{\Omega}=\left[\mathbf{V}(\xi)\otimes\mathbf{I}_N\right]^{-\frac{1}{2}}$$

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Indirect Least Square (3)

Step 1 Run equation-by-equation OLS on the reduced form

$$\Rightarrow \hat{\Pi} \text{ and } \mathrm{Var}(\hat{\Pi})$$

Unbiased but not the best for Π.

Step 2 Estimate β and Γ matrices from Π estimates if they are identified.

Indirect Least Square (4)

Step 3 Calculate

$$\hat{\Sigma} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1M} \\ \hat{\sigma}_{12} & \hat{\sigma}_{22} & \ddots & \hat{\sigma}_{2M} \\ \vdots & \ddots & \ddots & \vdots \\ \hat{\sigma}_{1M} & \hat{\sigma}_{2M} & \cdots & \hat{\sigma}_{MM} \end{bmatrix}$$

$$\hat{\mathbf{V}}(\xi) = [\mathrm{I} - \hat{\Gamma}^{T}]^{-1} \hat{\Sigma} \, [\mathrm{I} - \hat{\Gamma}]^{-1}$$

Indirect Least Square (5)

Step 4 Estimate the reduced form using SUR with Ω weighting matrix.

$$\hat{\boldsymbol{\Omega}} = \left[\hat{\mathbf{V}}(\boldsymbol{\xi}) \otimes \mathbf{I}_N\right]^{-\frac{1}{2}}$$

$$= \hat{\mathbf{V}}(\boldsymbol{\xi})^{-\frac{1}{2}} \otimes \mathbf{I}_N$$

For Feasible ILS (FILS), go back to Step 2 until convergence

Indirect Least Square (6)

Note that

1) if they are identified ILS estimators are just valid (consistent) but not the best

2) Calculation of Var($\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\Gamma}}$) from Var($\hat{\prod}$) is rather complicated as their relationship is non-linear.

Indirect Least Square (7)

3) FILS coincides with 2SLS with only X's as IV.

4) ILS may be non-unique (over-identified). No over-identification problem for 2SLS.

=> ILS is much less popular than 2SLS.

ID Problem Example (1)

Gujarati, p751

$$Y_{1t} = \gamma_{12}Y_{2t} + \gamma_{13}Y_{3t} + \beta_{11} + \beta_{12}X_{2t} + u_{1t}$$
$$Y_{2t} = \gamma_{23}Y_{3t} + \beta_{21} + \beta_{22}X_{2t} + \beta_{23}X_{3t} + u_{2t}$$
$$Y_{3t} = \gamma_{31}Y_{1t} + \beta_{31} + \beta_{32}X_{2t} + \beta_{33}X_{3t} + u_{3t}$$
$$Y_{4t} = \gamma_{41}Y_{1t} + \gamma_{42}Y_{2t} + \beta_{41} + \beta_{44}X_{4t} + u_{4t}$$

ID Problem Example (2)

Note that

- there are 16 parameters in (Γ, β) to be estimated.
- There are also 16 π's in Π matrix
- Can (Γ, β) be identified from Π matrix? It seems OK. In fact, it is not.

ID Problem Example (3)

<u>Matrix Form</u>

$$[Y_{1t} \quad Y_{2t} \quad Y_{3t} \quad Y_{4t} \quad 1 \quad X_{2t} \quad X_{3t} \quad X_{4t}]\begin{bmatrix} 1 & 0 & -\gamma_{31} & -\gamma_{41} \\ -\gamma_{12} & 1 & 0 & -\gamma_{42} \\ -\gamma_{13} & -\gamma_{23} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta_{11} & -\beta_{21} & -\beta_{31} & -\beta_{41} \\ -\beta_{12} & -\beta_{22} & -\beta_{32} & 0 \\ 0 & -\beta_{23} & -\beta_{33} & 0 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix} = \begin{bmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{bmatrix}$$

$$[\mathbf{Y}\ \mathbf{X}]\begin{bmatrix} \mathbf{I} - \boldsymbol{\Gamma} \\ -\boldsymbol{\beta} \end{bmatrix} = \mathbf{u}$$

U_1

ID Problem Example (4)

Equation#1

$$U_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\beta_{23} & -\beta_{33} & 0 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix}$$

$$m_1 + k_1 = 4 = K \text{ ==> } exact\,?$$

$$Rank(U_1) = 2 \text{ ==> } unidentified$$

ID Problem Example (5)

Equation#2

$$U_2 = \begin{bmatrix} 1 & 0 & -\gamma_{31} & -\gamma_{41} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix}$$

$$m_2 + k_2 = 4 = K \text{ ==> } exact\,?$$

$$Rank(U_2) = 2 \text{ ==> } unidentified$$

ID Problem Example (6)

Equation#3

$$U_3 = \begin{bmatrix} -\gamma_{12} & 1 & 0 & -\gamma_{42} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_{44} \end{bmatrix}$$

$$m_3 + k_3 = 4 = K ==> exact?$$

$$Rank(U_3) = 2 ==> unidentified$$

ID Problem Example (7)

Equation#4

$$U_4 = \begin{bmatrix} -\gamma_{13} & -\gamma_{23} & 1 & 0 \\ -\beta_{12} & -\beta_{22} & -\beta_{32} & 0 \\ 0 & -\beta_{23} & -\beta_{33} & 0 \end{bmatrix}$$

$$m_4 + k_4 = 4 = K ==> exact?$$

$$Rank(U_4) = 3 ==> identified$$

ID Problem Example (8)

2SLS

All four equations will be identified with X's as IV (no IV from outside)

ILS

Only eq#4 will be identified. The other three are not completely identified. At least one parameter in each of these three equations will be unidentified according to the rank of U_m sub-matrices.

ID Problem Example (9)

If equation#2 is subject to an within-equation restriction, e.g.,

$$\gamma_{23} + \beta_{23} = 1$$

$$m_2 + k_2 - l_2 = 3 < K$$

$$==> over-identified\,?$$

Still, $Rank(U_2) = 2 ==> unidentified$

ID Problem Example (10)

If equation#4 is subject to an within-equation restriction, e.g.,

$$\gamma_{41} = \beta_{42}$$

$$m_4 + k_4 - l_4 = 3 < K$$

$$==> over-identified\ ?$$

$$Rank(U_4) = 3 ==> over-identified$$

3SLS (1)

3SLS or Three-Stage Least Square is the simultaneous extension of 2SLS to take advantage of SUR technique to improve the estimation efficiency.

Note that in 2SLS the error terms are uncorrelated across equations. That is why equation-by-equation 2SLS is appropriate.

3SLS (2)

Start with the same $\mathbf{Z}_m$ as in 2SLS.

<u>Steps</u>

0) Set Σ=I

1) applying $\mathbf{Z}_m$ as IV, the solution to equation *m* is equivalent to that of

$$\mathbf{A}_m \mathbf{Y}_m = \mathbf{A}_m \mathbf{Y}_m \boldsymbol{\Gamma}_{m.} + \mathbf{A}_m \mathbf{X}\boldsymbol{\beta}_{m.} + \mathbf{A}_m \boldsymbol{\varepsilon}_m$$

Referred to IV section

3SLS (3)

2) Form stack (one-equation model)

$$\begin{bmatrix} A_1Y_{1.} \\ A_2Y_{2.} \\ | \\ A_MY_{M.} \end{bmatrix} = \begin{bmatrix} A_1Y\gamma_{1.}+A_1X\beta_{1.} & 0 & 0 & 0 \\ 0 & A_2Y\gamma_{2.}+A_2X\beta_{2.} & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_MY\gamma_{M.}+A_1X\beta_{M.} \end{bmatrix} + \underbrace{\begin{bmatrix} A_1\varepsilon_{1.} \\ A_2\varepsilon_{2.} \\ \\ A_M\varepsilon_{M.} \end{bmatrix}}_{e}$$

3SLS (4)

3) Estimate the weighting matrix Ω

$$\Omega = \begin{bmatrix} \sigma_{11}\mathbf{A}_{1.}\mathbf{A}_{1.} & \sigma_{21}\mathbf{A}_{2.}\mathbf{A}_{1.} & \cdots & \sigma_{M1}\mathbf{A}_{M.}\mathbf{A}_{1.} \\ \sigma_{12}\mathbf{A}_{1.}\mathbf{A}_{2.} & \sigma_{22}\mathbf{A}_{2.}\mathbf{A}_{2.} & \cdots & \sigma_{M2}\mathbf{A}_{M.}\mathbf{A}_{2.} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1M}\mathbf{A}_{1.}\mathbf{A}_{M.} & \sigma_{2M}\mathbf{A}_{2.}\mathbf{A}_{M.} & \cdots & \sigma_{MM}\mathbf{A}_{M.}\mathbf{A}_{M.} \end{bmatrix}^{-\frac{1}{2}}$$

3SLS (5)

3) Apply SUR to the stacked form using the estimated weight matrix.

==> valid estimates for γ,β and Σ

Note that there are restriction on Γ. Its diagonal elements are zero. Apply RLS.

Go back to 2) until convergence

GMM (1)

Generalized Methods of Moments

_extension over 3SLS by giving own weights and cross weights to each of IV to improve the estimation efficiency.

Same concept as GMM for one-equation model. Apply GMM to the stacked form which is now a single-equationed model.

GMM (2)

For each equation *m*, it is expected that

$$E(\mathbf{Z}_m(\mathbf{Y}_{m.}[I-\Gamma_{m.}]-X\beta_{m.}))=\mathbf{0}$$

for m=1,…,M

Moment Conditions

GMM (3)

For simplicity, assume that all the equations share the same set of IV

$\mathbf{Z}_m = \mathbf{Z} = [Z_1, Z_2, \ldots, Z_S]$

for all m=1,…,M

That is,

$E(\mathbf{Z}_s(\mathbf{Y}_m[I-\Gamma_{m.}]-X\beta_{m.}))=0$ for all s=1,…,S and m=1,…,M

GMM (4)

Sample Analogy

$$(\hat{\mathbf{\Gamma}}, \hat{\boldsymbol{\beta}})_{GMM} = \arg\min \sum_{\substack{m=1 \\ n=1}}^{M} \sum_{\substack{s=1 \\ t=1}}^{S} \left(w_{sm,tn} \mathbf{Z}_s^T (\mathbf{Y}_{m.} - \mathbf{X}\boldsymbol{\beta}_{m.}) \mathbf{Z}_t^T (\mathbf{Y}_{n.} - \mathbf{X}\boldsymbol{\beta}_{n.}) \right)$$

where $w_{sm,tn}$ = weight for the combination

GMM (5)

Put weights in the matrix form

$$\mathbf{W}_{mn} = \begin{bmatrix} w_{1m,1n} & w_{1m,2n} & & w_{1m,Sn} \\ w_{1m,2n} & w_{2m,2n} & & w_{2m,Sn} \\ & & \ddots & \\ w_{1m,Sn} & w_{2m,Sn} & & w_{Sm,Sn} \end{bmatrix}_{\text{SxS}}$$

GMM (6)

Put weights in the matrix form

$$\boldsymbol{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & & \mathbf{W}_{1M} \\ \mathbf{W}_{12} & \mathbf{W}_{22} & & \mathbf{W}_{2M} \\ & & \ddots & \\ \mathbf{W}_{1M} & \mathbf{W}_{2M} & & \mathbf{W}_{MM} \end{bmatrix}_{\text{SMxSM}}$$

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GMM (7)

Appropriate weight is

$$\hat{\boldsymbol{W}} = \left[\hat{\mathbf{V}}[\mathbf{Z}^T(\mathbf{Y}[\mathbf{I}-\hat{\boldsymbol{\Gamma}}]-\mathbf{X}\hat{\boldsymbol{\beta}})]\right]^{-1}$$

Estimation
Step 1 W=I (SMxSM identity matrix)
Step 2 Minimize

$$(\mathbf{Y}[\mathbf{I}-\boldsymbol{\Gamma}]-\mathbf{X}\boldsymbol{\beta})^T\mathbf{ZWZ}^T(\mathbf{Y}[\mathbf{I}-\boldsymbol{\Gamma}]-\mathbf{X}\boldsymbol{\beta})$$

GMM (8)

Step 3 Estimate W for next iteration

$$\hat{\mathbf{W}} = \boldsymbol{\Sigma}^{-1}$$

$$= \frac{1}{N}(\mathbf{Y}[\mathbf{I}-\hat{\boldsymbol{\Gamma}}]-\mathbf{X}\hat{\boldsymbol{\beta}})^T(\mathbf{Y}[\mathbf{I}-\hat{\boldsymbol{\Gamma}}]-\mathbf{X}\hat{\boldsymbol{\beta}})\otimes\mathbf{Z}^T\mathbf{Z}$$

Back to Step 2 until convergence occurs