Heteroscedasticity

Covered Topics

- What is Heteroscedasticity?
- Weighted Least Square (WLS)
- Tests for Variance equation
- Remedies
- Generalized Heteroscedasticity

What is Heteroscedasticity?

Violation of constant variance of ε_i's but they are still independent.

$$\mathrm{V}(\varepsilon_i) = \sigma_i^2$$

$$\mathrm{COV}(\varepsilon_i, \varepsilon_j) = 0, \forall i \neq j$$

$$\mathbf{V}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n^2 \end{bmatrix}$$

The error term (ε) is said to be heteroscedastic.

Weighted Least Square (1)

What happened to OLS estimators?

==> OLS is still LUE but not BLUE.

==> Large sample properties ???

Weighted Least Square (WLS)

$$w_i Y_i = \beta_1 w_i X_{1i} + \beta_2 w_i X_{2i} + \ldots + \beta_K w_i X_{Ki} + w_i \varepsilon_i$$

$$\underline{Y}_i = \beta_1 \underline{X}_{1i} + \beta_2 \underline{X}_{2i} + \ldots + \beta_K \underline{X}_{Ki} + \nu_i$$

where

$$w_i = \frac{1}{\sigma_i} \quad \underline{Y}_i = w_i Y_i \quad \underline{X}_{ki} = w_i X_{ki}, k = 1,\ldots,K \quad \nu_i = w_i \varepsilon_i$$

Weighted Least Square (2)

Matrix form

$$\underline{\mathbf{Y}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{Y},\ \underline{\mathbf{X}} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{X}, \boldsymbol{\nu} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\Sigma}^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_n} \end{bmatrix}$$

Weighted Least Square (3)

Note that $\underline{\mathbf{Y}} = \underline{\mathbf{X}}\boldsymbol{\beta} + \underline{\boldsymbol{\varepsilon}}$ is CLNRM and

$\boldsymbol{\nu}$ is homoscedastic.

$$\mathrm{V}(\nu_i) = 1, \forall i = 1,\dots,n$$

$$\mathbf{V}(\boldsymbol{\nu}) = \mathbf{I}_n$$

WLS estimator $\hat{\boldsymbol{\beta}} = \left[\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{X}}\right]^{-1}\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{Y}}$

$$\mathrm{V}(\hat{\boldsymbol{\beta}}) = \left[\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{X}}\right]^{-1}$$

Note that, given Σ, the estimator $\hat{\boldsymbol{\beta}}$ is BLUE.

Variance Equation (1)

Explanatory variables

- observation index *i*
- some or all the **X**'s
- conditional mean of Y given X
- variable *Z*'s not in the model
- lagged variables (to be discussed in Time Series Analysis)

Variance Equation (2)

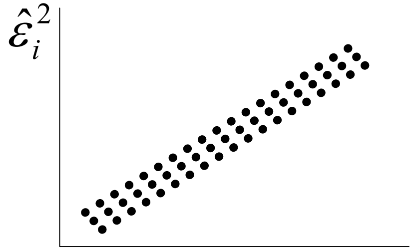
Detection

- squared residual plots
- White's Test
- Other LM tests

Squared Residual Plots (1)

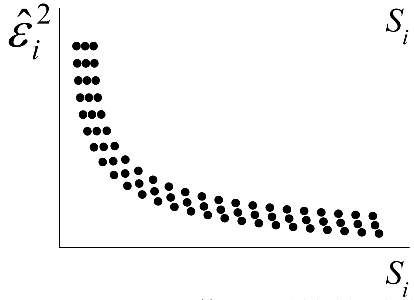
- squared residuals against the suspected explanatory variable (S)
- test against only one variable at a time
- functional forms, e.g., linear, log-lin, log-log, quadratic, polynomial, reciprocal, etc.

Squared Residual Plots (2)



Linear

$$V(\varepsilon_i) = \gamma_1 + \gamma_2 S_i$$



Reciprocal

$$V(\varepsilon_i) = \gamma_1 + \gamma_2 \frac{1}{S_i}$$

or log-lin

$$\ln(V(\varepsilon_i)) = \gamma_1 + \gamma_2 S_i$$

Tests for Heteroscedasticity (1)

Functional form of the variance equation must be assumed in most tests. For example,

- Breusch-Pagan's test assumes a linear function of suspected variables
- Glesjer's test sets the form of standard deviation σ_i instead of the variance
- **White**'s test adds squared and cross terms to the variance equation of BP test.

Tests for Heteroscedasticity (2)

- **Harvey-Godfrey**'s test assumes a log-lin or
  $$V(\varepsilon_i) = \exp(\gamma_1 + \gamma_2 S_{2i} + \gamma_3 S_{3i} + .. + \gamma_P S_{Pi})$$
  Note that HG assures positive fitted variances while others do not.
- Park test assumes double log form.
- Goldfeld-Quandt does not assume the form of the variance function. Instead, it checks for equality of the variances between the high group and the low group using variance ratio test(F-test). See text.

White's Test (1)

White's General Heteroscedasticity Test

<u>Concept</u>: The variables that are suspected to cause heteroscedasticity are the X's, their squared terms and their cross terms.

$$\begin{aligned} V(\varepsilon_i) = {} & \gamma_1 + \gamma_2 X_{2i} + .. + \gamma_K X_{Ki} \\ & + \gamma_{22} X_{2i}^2 + \gamma_{23} X_{2i} X_{3i} + .. + \gamma_{2K} X_{2i} X_{Ki} \\ & + \gamma_{33} X_{3i}^2 + \gamma_{34} X_{3i} X_{4i} + .. + \gamma_{3K} X_{3i} X_{Ki} \\ & \ddots \\ & + \gamma_{KK} X_{Ki}^2 + \varepsilon_i \end{aligned}$$

White's Test (2)

Run OLS on the auxiliary regression of the squared residuals and all the suspected variables above. Assume $X_{1i} = 0$ for all i.

Perform the overall F-test or LM test for the auxiliary regression run on the following hypothesis.

$$H_0 : \gamma_2 = \gamma_3 = ... = \gamma_K = \gamma_{22} = \gamma_{23} = ... = \gamma_{2K} = ... = \gamma_{KK} = 0$$

$$H_1 : \gamma_2 \neq \gamma_3 \neq ... \neq \gamma_K \neq \gamma_{22} \neq \gamma_{23} \neq ... \neq \gamma_{2K} \neq .... \neq \gamma_{KK} \neq 0$$

EViews also provide a quick solution for White's test.

White's Test (3)

Accept H_0 ==> No heteroscedasticity

If reject H_0, no specific remedy can be done.

Without any remedy, $\widehat{V(\hat{\boldsymbol{\beta}})}$ must be re-calculated since the old one is inconsistent (wrong). White's Heteroscedasticity Consistent Covariance is

$$\widehat{V(\hat{\boldsymbol{\beta}})} = \frac{n}{n-k} \left[\mathbf{X}^{\mathrm{T}}\mathbf{X}\right]^{-1} \left[\sum_{i=1}^{n} \hat{\varepsilon}_i^2 \mathbf{X}_i^{\mathrm{T}} \mathbf{X}_i\right] \left[\mathbf{X}^{\mathrm{T}}\mathbf{X}\right]^{-1}$$

where X_i is the i^{th} observation of X.

Generalized Remedies (1)

<u>F</u>easible <u>G</u>eneralized <u>L</u>east <u>S</u>quares (FGLS)

- Estimate the variance equation using auxiliary regression of squared residuals on the explanatory variables (for the variance equation), e.g.,

$$\hat{\varepsilon}_i^2 = V(\varepsilon_i) + \xi_i = \gamma_1 + \gamma_2 \frac{1}{S_i} + \xi_i$$

where ξ_i is the error term for the variance equation

Generalized Remedies (2)

- Calculate the fitted value of squared residuals or estimate $V(\varepsilon_i)$

$$\widehat{V(\varepsilon_i)} = \widehat{\hat{\varepsilon}_i^2} = \hat{\gamma}_1 + \hat{\gamma}_2 \frac{1}{S_i}$$

- Use the weight $w_i = \frac{1}{\sqrt{\widehat{V(\varepsilon_i)}}}$ in WLS

- FGLS is biased but consistent.

Special Cases (1)

Variance equation with single parameter, e.g.,

Example 1 $V(\varepsilon_i) = \sigma^2 \frac{1}{S_i}$ or $\hat{\varepsilon}_i^2 = \sigma^2 \frac{1}{S_i} + \xi_i$

where σ^2 is a positive constant.

Apply WLS with $w_i = \sqrt{S_i}$

Note that $V(V_i) = \sigma^2$ for all $i=1,\ldots,n$

==> WLS estimator is unbiased. No FGLS needed

Special Cases (2)

Example 2 $V(\varepsilon_i) = \sigma^2 (S_i)^2$

$$\hat{\varepsilon}_i^2 = \sigma^2 (S_i)^2 + \xi_i$$

where σ^2 is a positive constant.

Apply WLS with $w_i = 1/S_i$

Note that $V(V_i) = \sigma^2$ for all $i=1,\ldots,n$

==> WLS estimator is unbiased. No FGLS needed.

What is Generalized Heteroscedasticity?

Violation of constant variance and/or independence of ε_i's

$$V(\varepsilon_i) = \sigma_i^2$$

$$COV(\varepsilon_i, \varepsilon_j) = \sigma_{ij} \neq 0 \text{ for some } i, j$$

$$\mathbf{V(\varepsilon)} = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_{n-1,n} \\ \sigma_{1n} & \cdots & \sigma_{n-1,n} & \sigma_n^2 \end{bmatrix}$$

The error term (ε) is said to be generalized heteroscedastic.

Generalized Least Square (1)

OLS or WLS estimators are still LUE but not BLUE.

Generalized Least Square (GLS)

$$\underline{\mathbf{Y}} = \mathbf{WY}, \quad \underline{\mathbf{X}} = \mathbf{WX}, \boldsymbol{\nu} = \mathbf{W}\boldsymbol{\varepsilon}$$

where $\mathbf{W}$ is a nxn symmetric matrix such that

$$\boldsymbol{\Sigma}^{-1} = \mathbf{WW}$$

Generalized Least Square (2)

Note that $\underline{\mathbf{Y}} = \underline{\mathbf{X}}\boldsymbol{\beta} + \boldsymbol{\nu}$ is CLNRM and

$\boldsymbol{\nu}$ is homoscedastic. $\mathrm{V}(\nu_i) = 1, \forall i = 1,\ldots,n$

$$\mathbf{V}(\boldsymbol{\nu}) = \mathbf{I}_n$$

GLS estimator $\hat{\boldsymbol{\beta}} = \left[\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{X}}\right]^{-1}\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{Y}}$

$$\mathrm{V}(\hat{\boldsymbol{\beta}}) = \left[\underline{\mathbf{X}}^{\mathrm{T}}\underline{\mathbf{X}}\right]^{-1}$$

Note that if $\mathrm{V}(\varepsilon)$ is known, $\mathbf{W}$ can be calculated.

So is $\hat{\boldsymbol{\beta}}$. ==>the estimator $\hat{\boldsymbol{\beta}}$ is BLUE.

Generalized Least Square (3)

If Σ is known up to a proportion,

$$\boldsymbol{\Sigma} = \sigma^2\boldsymbol{\Omega} = \sigma^2\begin{bmatrix} \phi_1^2 & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{12} & \phi_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \phi_{n-1,n} \\ \phi_{1n} & \cdots & \phi_{n-1,n} & \phi_n^2 \end{bmatrix}$$

where σ^2 is unknown but Ω matrix is known.

Generalized Least Square (4)

Choose the weighting matrix $\mathbf{W}$ such that

$$\mathbf{WW} = \boldsymbol{\Omega}^{-1}$$

Define $\underline{\mathbf{Y}} = \mathbf{WY}, \quad \underline{\mathbf{X}} = \mathbf{WX}, \boldsymbol{\nu} = \mathbf{W}\boldsymbol{\varepsilon}$

Note that $\underline{\mathbf{Y}} = \underline{\mathbf{X}}\boldsymbol{\beta} + \boldsymbol{\nu}$ is CLNRM and

$\boldsymbol{\nu}$ is homoscedastic or $\mathbf{V}(\boldsymbol{\nu}) = \sigma^2\mathbf{I}_n$

==> GLS estimator is BLUE

Generalized Least Square (5)

$\mathrm{V}(\boldsymbol{\varepsilon})$ is restricted in parameters. For example,

$$\mathbf{V}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}\mathbf{P}\boldsymbol{\Sigma}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_{n-1,n} \\ \rho_{1n} & \cdots & \rho_{n-1,n} & 1 \end{bmatrix}$$