Introduction to Time Series Models

TS Data Analysis

- Graphical (vs time)
  - –level or change
  - –single or multiple
- Numerical
  - –Decomposition
  - –Generating Process

Covered Topics

- Decomposition Approach (only introduction)
- Generating Process Approach (main topic)
- Difference Equation

Definition

Define Y as a time series

Y_t is the observed value of Y at time t. Note that Y_t is a random variable

t is used as the observation index (in place of i) which indicates the chronological order.

Components of TS (1)

- Time trend as function of t (Tr_t)
- Seasonal (Sn_t)
- Cyclical (Cl_t)
- Irregular or random (Ir_t)

Not to be discussed in details

Components of TS (2)

Some patterns of time trend (Tr_t)

- linear
- polynomial
- lin-log, log-lin, log-log
- reciprocal
- logistic

Components of TS (3)

Modelling the TS components

- Additive

$$Y_t = Tr_t + Sn_t + Cl_t + Ir_t$$

- Multiplicative

$$Y_t = Tr_t * Sn_t * Cl_t * Ir_t$$

Decomposition Method (1)

<u>Steps</u>

- ignore the Cyclical component
- identify and <u>separate</u> the trend (de-trend)
- identify and <u>separate</u> seasonal component from the de-trended data

Decomposition Method (2)

To separate components, use averaging and/or regression and/or smoothing technique

To forecast Y, forecast each component and, then, add or multiply them, depending on the model selected.

Smoothing Non-seasonal

- Single-exponential Smoothing
- Double-exponential Smoothing
- Triple-exponential Smoothing
- Holt-Winters's exponential Smoothing (2 smoothing constants)

Smoothing Seasonal

- Holt-Winters's exponential Smoothing (3 smoothing constants)

See EViews for more explanation.

Note that Eviews does not support triple-smoothing.

TS Generating Processes

- Linear in t, Y_1, Y_2,…, Y_t,… and random components
- Non-linear in t, Y_1, Y_2,…, Y_t, … and random components

Linear Generating Processes

- White noise
- Auto-regressive (AR)
- Moving average (MA)
- Mixed AR and MA (ARMA)
- Differencing (ARIMA)
- ARIMAX (ARIMA with X)

White Noise Processes

Note that Y is a white noise process if

$$E(Y_t) = 0 \quad \text{for } \forall t$$

$$V(Y_t) = \sigma^2 \quad \text{for } \forall t$$

$$Cov(Y_t, Y_s) = 0 \quad \text{for } \forall t \neq s$$

That is, $Y_t = \varepsilon_t$.

AR Processes

Basic Form of AR(p) or AR process of order p

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t$$

where ε_t is a white noise.

That is, Y_t. is a linear function of its own lagged values plus a random component.

MA Processes

Basic Form of MA(q) or MA process of order q

$$Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

where ε's are white noises

That is, Y_t. is a moving average of the current and lagged values of white noses.

ARMA Processes

Basic Form of ARMA(p,q)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p}$$
$$+ \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$
$$+ \cdots + \theta_q \varepsilon_{t-q}$$

Note that

AR(p)=ARMA(p,0)

MA(q) = ARMA(0,q)

AR(0)=MA(0)=white noise

Higher Differences

2nd difference

$$\Delta^2 Y_t \equiv \Delta Y_t - \Delta Y_{t-1}$$
$$\equiv Y_t - 2Y_{t-1} + Y_{t-2}$$

dth difference

$$\Delta^d Y_t \equiv \Delta(\Delta^{d-1} Y_t)$$
$$\equiv \Delta^{d-1} Y_t - \Delta^{d-1} Y_{t-1}$$

Note that $\Delta^0 Y_t \equiv Y_t$

Seasonal Differences

If there are *S* seasons in a year,

1st seasonal difference

$$\Delta_S Y_t = Y_t - Y_{t-S}$$

Note that $\Delta_4 Y_t \neq \Delta^4 Y_t$

dth seasonal difference

$$\Delta_S^d Y_t = \Delta_S^d (\Delta_S^{d-1} Y_t)$$
$$= \Delta_S^{d-1} Y_t - \Delta_S^{d-1} Y_{t-S}$$

ARIMA Processes

Basic Form of ARIMA(p,d,q)

$$\Delta^d Y_t = \phi_1 \Delta^d Y_{t-1} + \phi_2 \Delta^d Y_{t-2} + \cdots + \phi_p \Delta^d Y_{t-p}$$
$$+ \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

I stands for Integrated. d is called the integrated order.

Note that ARIMA(p,0,q)=ARMA(p,q)

ARIMAX Processes

$$
\begin{aligned}
\Delta^d Y_t = {} & \beta_1 X_{1t} + \cdots + \beta_K X_{Kt} \\
& + \phi_1 \Delta^d Y_{t-1} + \phi_2 \Delta^d Y_{t-2} \\
& + \cdots + \phi_p \Delta^d Y_{t-p} \\
& + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}
\end{aligned}
$$

Note that some of the X's could be a constant or time trend or dummy variables.

Key Issues

- Parameter Estimation
  - –Is LS valid?
  - –Any additional assumption?
- Their Standard errors
- Predict Y_{t+i}

Difference Equation

- Discrete analogy to differential equations
- Describe how a quantitative variable (Y) is related to its past values and exogenous series (u)
- Basis for a time series model

General Form

$$F(\Delta^p Y_t, \Delta^{p-1} Y_{t-1}, \ldots, \Delta^0 Y_{t-p}, x_t) = 0$$

or $F(\Delta^p Y_t, \Delta^{p-1} Y_t, \ldots, \Delta^0 Y_t, x_t) = 0$

Linear Form

$$
\begin{aligned}
\Delta^p Y_t + \beta_1 \Delta^{p-1} Y_{t-1} + \beta_2 \Delta^{p-2} Y_{t-2} \\
+ \cdots + \beta_p \Delta^0 Y_{t-p} = x_t
\end{aligned}
$$

More Preferable Form

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + x_t$$

where $\phi_1 = p - \beta_1$

$$\phi_2 = -\frac{1}{2}p(p-1) + (p-1)\beta_1 - \beta_2$$

$$\vdots$$

Lag Operator

$$LY_t = Y_{t-1}$$

$$L^2 Y_t = L(LY_t) = L(Y_{t-1}) = Y_{t-2}$$

$$\vdots$$

$$L^p Y_t = L(L^{p-1} Y_t) = L(Y_{t-(p-1)}) = Y_{t-p}$$

Basic algebra tool for TS process analyses

Alternative Form

Using Lag Operator

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) Y_t = x_t$$

$$(1 - \lambda_1 L)(1 - \lambda_2 L) \ldots (1 - \lambda_p L) Y_t = x_t$$

where λ's are p characteristic roots of

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_p \lambda^0 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_p) = 0$$

Solve Diff. Eqn

Determine the values of

$$Y_t, Y_{t-1}, Y_{t-2}, \ldots$$

in terms of $\phi_1, \cdots, \phi_p, t, u_t, u_{t-1}, \ldots$

that satisfies the difference equation

Note that u could be constant, time trend or another time series

Solution (1)

Homogenous Solution u_t=0

$$Y_t^h = \sum_{j=1}^{p} A_j \lambda_j^t$$

Particular Solution $Y_t = Y_{t-1}$

$$Y_t^P = \frac{1}{1-\phi_1-\phi_2-\cdots-\phi_p} x_t$$

Solution (2)

General Solution

$$Y_t = Y_t^h + Y_t^p$$

$$= \sum_{j=1}^{p} A_j \lambda_j^t + \frac{1}{1-\phi_1-\phi_2-\cdots-\phi_p} x_t$$

Solution (3)

Homo. solution is convergent or stable if (exact conditions)

$$\left|\lambda_j\right| < 1 \text{ for } \forall j$$

=> Y_t^h converges to zero

Non-convergent if

$$\left|\lambda_j\right| \geq 1 \text{ for some } j$$

Solution (4)

Necessary conditions

$$\sum_{j=1}^{p} \phi_j < 1$$

Sufficient Conditions

$$\sum_{j=1}^{p} \left|\phi_j\right| < 1$$

Schur Theorem

Schur Theorem (1)

$$\Delta_1 = \begin{vmatrix} 1 & -\phi_p \\ -\phi_p & 1 \end{vmatrix} > 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -\phi_p & -\phi_{p-1} \\ -\phi_1 & 1 & 0 & -\phi_p \\ -\phi_p & 0 & 1 & -\phi_1 \\ -\phi_{p-1} & -\phi_p & 0 & 1 \end{vmatrix} > 0$$

Schur Theorem (2)

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 0 & -\phi_p & -\phi_{p-1} & -\phi_{p-2} \\ -\phi_1 & 1 & 0 & 0 & -\phi_p & -\phi_{p-1} \\ -\phi_2 & -\phi_1 & 1 & 0 & 0 & -\phi_p \\ -\phi_p & 0 & 0 & 1 & -\phi_1 & \\ -\phi_{p-1} & -\phi_p & 0 & 0 & 1 & -\phi_1 \\ -\phi_{p-2} & -\phi_{p-1} & -\phi_p & 0 & 0 & 1 \end{vmatrix} > 0$$

Schur Theorem (3)

$$\Delta_p = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & -\phi_p & -\phi_{p-1} & \cdots & -\phi_2 & -\phi_1 \\ & 1 & \ddots & \ddots & 0 & 0 & -\phi_p & \ddots & \ddots & -\phi_2 \\ & & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 0 & 0 & \ddots & \ddots & -\phi_p & -\phi_{p-1} \\ & & & & 1 & 0 & 0 & \cdots & 0 & -\phi_p \\ & & & & & 1 & & & & \\ & & & & & 0 & 1 & & & \\ & & & & & \vdots & \ddots & \ddots & & \\ & & & & & 0 & \ddots & \ddots & 1 & \\ & & & & & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} > 0$$