Stationary Processes

Covered Topics

- Definition of Stationarity
- Types of Time Series Generating Processes
- Estimation
  - Box-Jenkins Method

Types of Stationarity

- Weak
- Strong
- Stochastic

Weak Stationarity

Y ~ Weakly Stationary if

- Mean stationary $E(Y_t)=\mu < \infty$ for all t
  ==>time-invariant mean
- Variance stationary $V(Y_t) =\gamma_0 < \infty$ for all t
  ==>time-invariant variance
- Covariance stationary
  $Cov(Y_t,Y_{t-s})=\gamma_s < \infty$ for all t,s
  ==>time-invariant but spread-specific

Strong Stationarity

<u>Strongly Stationary Process</u>

- require only Covariance Stationarity

Stochastic Stationarity

Y ~ Stochastically Stationary if

pdf(…,Y_t=a,Y_{t+1}=b,…)

= pdf(…,Y_s=a,Y_{s+1}=b,…) for all *t*,*s*

Note that

Weak =>Strong

Stochastic =>Weak

Why is Stationarity important?

OLS involving a time series data requires additional assumptions. One of them is “stationarity” of the error terms.

Otherwise, OLS will be invalid.

Stationary MA(q)

$$V(Y_t) = \sigma^2(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2)$$

$$Cov(Y_t, Y_{t-1}) = \sigma^2\left[\theta_1+\theta_1\theta_2+\theta_2\theta_3+\cdots+\theta_{q-1}\theta_q\right]$$

$$Cov(Y_t, Y_{t-2}) = \sigma^2\left[\theta_2+\theta_1\theta_3+\theta_2\theta_4+\cdots+\theta_{q-2}\theta_q\right]$$

$$\vdots$$

$$Cov(Y_t, Y_{t-q}) = \sigma^2\left[\theta_q\right]$$

$$Cov(Y_t, Y_{t-j}) = 0 \quad \text{for } j > q$$

Given finite order *q*, pure MA(*q*) is always stationary.

Stationary MA(∞)

However, stationarity cannot be guaranteed for an MA process with infinite order.

Even convergence of θ_q or

$$\theta_q -> 0 \quad \text{as} \quad q -> \infty$$

does not imply stationarity.

Stationary AR(1) ----(1)

Given $\quad Y_t = \mu + \phi_1 Y_{t-1} + \varepsilon_t$

$$\begin{aligned} Y_t &= \mu + \phi_1(\mu + \phi_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \mu + \phi_1\mu + \phi_1^2 Y_{t-2} + \phi_1\varepsilon_{t-1} + \varepsilon_t \\ &= \mu + \phi_1\mu + \phi_1^2\mu + \cdots \\ &\quad \underbrace{+\cdots + \phi_1^2\varepsilon_{t-2} + \phi_1\varepsilon_{t-1} + \varepsilon_t}_{\text{MA}(\infty)} \end{aligned}$$

Stationary AR(1) ----(2)

$$\begin{aligned} E(Y_t) &= \mu + \phi_1\mu + \phi_1^2\mu + \cdots \\ &\quad + \cdots + \phi_1^2 E(\varepsilon_{t-2}) + \phi_1 E(\varepsilon_{t-1}) + E(\varepsilon_t) \\ &= \mu + \phi_1\mu + \phi_1^2\mu + \cdots \\ &= \frac{\mu}{1-\phi_1} \qquad \text{if } |\phi_1| < 1 \end{aligned}$$

==> Mean-stationary

Stationary AR(1) ----(3)

$$\begin{aligned} V(Y_t) &= \cdots + \phi_1^4 V(\varepsilon_{t-2}) + \phi_1^2 V(\varepsilon_{t-1}) + V(\varepsilon_t) \\ &= \cdots + \phi_1^4\sigma^2 + \phi_1^2\sigma^2 + \sigma^2 \\ &= \frac{\sigma^2}{1-\phi_1^2} \qquad \text{if } |\phi_1| < 1 \end{aligned}$$

==> Variance-stationary

Stationary AR(1) ----(4)

$$Cov(Y_t, Y_{t-1}) = \cdots + \phi_1^3 V(\varepsilon_{t-2}) + \phi_1 V(\varepsilon_{t-1})$$

$$= \cdots + \phi_1^3 \sigma^2 + \phi_1 \sigma^2$$

$$= \frac{\sigma^2}{1-\phi_1^2}\phi_1 \qquad \text{if } |\phi_1| < 1$$

Stationary AR(1) ----(5)

$$Cov(Y_t, Y_{t-j}) = \cdots + \phi_1^{j+2} V(\varepsilon_{t-j}) + \phi_1^j V(\varepsilon_{t-j})$$

$$= \cdots + \phi_1^{j+2}\sigma^2 + \phi_1^j \sigma^2$$

$$= \frac{\sigma^2}{1-\phi_1^2}\phi_1^j \qquad \text{if } |\phi_1| < 1$$

==> Covariance-stationary

==> also converge to zero

Stationary AR(1) ----(6)

With linear time trend

$$Y_t = \mu + \delta t + \phi_1 Y_{t-1} + \varepsilon_t$$

$$Y_t = \mu + \delta t + \phi_1(\mu + \delta(t-1) + \phi_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$= \mu + \phi_1\mu + \phi_1^2\mu + \cdots$$

$$-(\phi_1 + 2\phi_1^2 + 3\phi_1^3 + \cdots)\delta$$

$$+(1 + \phi_1 + \phi_1^2 + \cdots)\delta t$$

$$\underbrace{+\cdots + \phi_1^2\varepsilon_{t-2} + \phi_1\varepsilon_{t-1} + \varepsilon_t}_{\text{MA}(\infty)}$$

Stationary AR(1) ----(7)

For $|\phi_1| < 1$

$$E(Y_t) = \frac{\mu}{1-\phi_1} - \frac{\phi_1\delta}{(1-\phi_1)^2} + \frac{\delta}{1-\phi_1}t$$

==>mean non-stationary

but still variance and

covariance-stationary

Stationary AR(1) ----(8)

Note that AR(1) with linear time trend is equivalent to

$$Y_t - \frac{\delta}{1-\phi_1}t = \mu + \delta - \frac{\delta}{1-\phi_1} + \phi_1\left(Y_{t-1} - \frac{\delta}{1-\phi_1}(t-1)\right) + \varepsilon_t$$

Stationary AR(1) ----(9)

Can prove that the de-trended *Y* or

$$Z_t = Y_t - \frac{\delta}{1-\phi_1}t$$

is weakly stationary.

==> *Y* is defined as <u>trend-stationary</u>

Stationary AR(1) ----(10)

Note that

$$AR(1) ===> MA(\infty)$$

but $$AR(1) \;\cancel{<===}\; MA(\infty)$$

Random Walk (1)

AR(1) with $\phi_1 = 1$

Without drift

$$Y_t = Y_{t-1} + \varepsilon_t \quad \text{or} \quad \Delta Y_t = \varepsilon_t$$

With drift (intercept or constant term)

$$\Delta Y_t = \mu + \varepsilon_t$$

where $\Delta Y_t = Y_t - Y_{t-1}$

or the first difference of Y_t

Random Walk (2)

AR(1) with $\phi_1 = 1$

Note that a random walk process is non-stationary but its first difference is.

Y is also referred to as <u>first-difference stationary</u>.

Stationary ARMA(1,q)

$$Y_t = \phi_1 Y_{t-1} + u_t$$

where u_t follows a stationary MA process with auto-covariance

$$\psi_j = Cov(u_t, u_{t-j})$$

Given $|\phi_1|<1$, Y is stationary if ψ_j converges to zero

Stationary AR(p) ----(1)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t$$

$$Y_t = \phi_1 L Y_t + \phi_2 L^2 Y_t + \cdots + \phi_p L^p Y_t + \varepsilon_t$$

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) Y_t = \varepsilon_t$$

Stationary AR(p) ----(2)

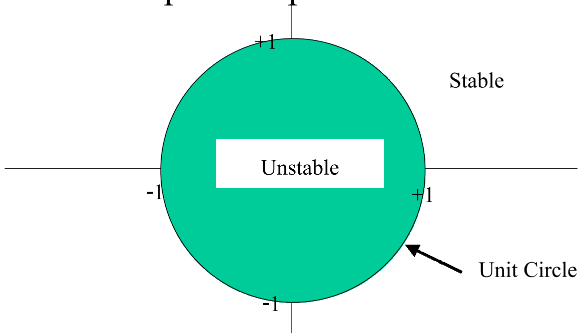
Polynomial Characteristic function of AR(p)

$$1 - \phi_1 \lambda - \phi_2 \lambda^2 - \cdots - \phi_p \lambda^p = 0$$

<u>Stability Condition</u>: Y is stable if all the *p* roots of the above polynomial function (or AR roots) must lie outside the unit circle. Why? If not, Y will diverge away from its mean

Stationary AR(p) ----(3)

Note that a root could be real, complex and it could be unique or repeated.



Stationary AR(p) ----(4)

For p=1, $$1-\phi_1\lambda = 0$$

The only root is $$\lambda = \frac{1}{\phi_1}$$

Stability Condition requires that

$$|\lambda| > 1 \ \Longrightarrow \ |\phi_1| < 1$$

For AR(1), Stability cond. = Stationarity

Stationary AR(p) ----(5)

Alternative Polynomial Characteristic function of AR(p)

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \ldots - \phi_p = 0$$

Stability Condition : all the p roots of the above polynomial function (or inverted AR roots) must lie inside the unit circle

Stationary AR(p) ----(6)

$$(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L)Y_t = \varepsilon_t$$

Given that λ_1 is inside the unit circle,

$$(1-\lambda_2 L)\cdots(1-\lambda_p L)Y_t = (1-\lambda_1 L)^{-1}\varepsilon_t$$

is stationary.

Given that all the p AR roots are inside the unit circle, Y is stationary

AR(1)

Stationary ARMA(p,q)

Stability condition is only a <u>necessary</u> condition for stationarity condition.

Need additional conditions about MA part

Stationary ARMA(p,q)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q}$$

$$Y_t = \phi_1 L Y_t + \phi_2 L^2 Y_t \ldots + \phi_p L^p Y_t + \varepsilon_t + \theta_1 L \varepsilon_t + \theta_2 L^2 \varepsilon_t + \ldots + \theta_q L^q \varepsilon_t$$

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) \varepsilon_t$$

Stationary ARMA(p,q)

Polynomial Characteristic function for AR part

$$1 - \phi_1 \lambda - \phi_2 \lambda^2 - \ldots - \phi_p \lambda^p = 0$$

Polynomial Characteristic function for MA part

$$1 + \theta_1 \lambda + \theta_2 \lambda^2 + \ldots + \theta_q \lambda^q = 0$$

Stationary ARMA(p,q)

<u>Invertibility Conditions</u> :

==> all the *q* roots of the MA polynomial function must lie outside the unit circle

Stability of AR + Invertibility of MA

===>Stationarity of ARMA

Stationary ARMA(p,q)

Alternative Polynomial Characteristic function of MA part

$$\lambda^{q} - \theta_1 \lambda^{q-1} - \theta_2 \lambda^{q-2} - \ldots - \theta_q = 0$$

Invertibility Condition : all the q roots of the above polynomial function (or inverted MA roots) must lie inside the unit circle

Box-Jenkins Method (1)

Major Assumptions

- Y has no trend
- Y is generated by an ARIMA process with unknown (p,d,q)

Box-Jenkins Method (2)

Steps

- de-trending
- identification
- estimation
- diagnosis checking

Box-Jenkins Method (3)

De-trending

- run OLS of Y_t against t

$$Y_t = \alpha + \beta t + \varepsilon_t$$

- use $Y_t^D = Y_t - \hat{\alpha} - \hat{\beta} t$ in the next step

Box-Jenkins Method (4)

Identification

- set d=0
- plot sample correlogram of de-trended $\Delta^d Y_t$
- What is correlogram?
  - –Plot of ACF (AC vs time)
  - –Plot of PACF (PACF vs time)

Box-Jenkins Method (5)

Population Auto-Correlation

$$\rho_s = \frac{\gamma_s}{\gamma_0}, s = 1, 2, \ldots$$

$$\rho_0 = 1$$

which is population correlation between Y_t and Y_{t-s}.

All ρ_s's are unknown parameters.

Box-Jenkins Method (6)

Sample Auto-Covariance

$$\hat{\gamma}_s = \frac{1}{n}\sum_{t=1}^{n-s}\left(Y_t^D Y_{t-s}^D\right), s = 0, 1, 2, \ldots$$

which is estimator of γ_s.

Sample Auto-Correlation

$$\hat{\rho}_s = \frac{\hat{\gamma}_s}{\hat{\gamma}_0}, s = 1, 2, \ldots$$

which is estimator of ρ_s.

Box-Jenkins Method (7)

Population Partial Auto-Correlation

ρ_{ss} = correlation between Y_t and Y_{t-s}, ignoring the effects of Y's in between (if any)

Note that $\rho_{11} = \rho_1$

Box-Jenkins Method (8)

Formula for ρ_{ss}

$$\rho_{ss} = \frac{\rho_s - \sum_{j=1}^{s-1} \rho_{s-1,j}\rho_{s-j}}{1 - \sum_{j=1}^{s-1} \rho_{s-1,j}\rho_j}, s = 1, 2, \ldots$$

$$\rho_{sj} = \rho_{s-1,j} - \rho_{ss}\rho_{s-1,s-j}, j = 1, .., s-1$$

Need proof?

Box-Jenkins Method (9)

Sample Analogy

$$\hat{\rho}_{11} = \hat{\rho}_1$$

$$\hat{\rho}_{22} = \frac{\hat{\rho}_2 - \hat{\rho}_{11}\hat{\rho}_1}{1 - \hat{\rho}_{11}\hat{\rho}_1}$$

$$\hat{\rho}_{21} = \hat{\rho}_{11} - \rho_{22}\rho_{11}$$

$$\hat{\rho}_{33} = \frac{\hat{\rho}_3 - \hat{\rho}_{21}\hat{\rho}_2 - \hat{\rho}_{22}\hat{\rho}_1}{1 - \hat{\rho}_{21}\hat{\rho}_1 - \hat{\rho}_{22}\hat{\rho}_2}$$

$$\hat{\rho}_{32} = \hat{\rho}_{22} - \hat{\rho}_{33}\hat{\rho}_{21}$$

$$\hat{\rho}_{31} = \hat{\rho}_{21} - \hat{\rho}_{33}\hat{\rho}_{22}$$

Box-Jenkins Method (10)

Rule

Y is stationary if both ACF and PACF converge to zero.

Sample Analogy

Y is believed to be stationary if both ACF and PACF converge to "zero" region.

Box-Jenkins Method (10)

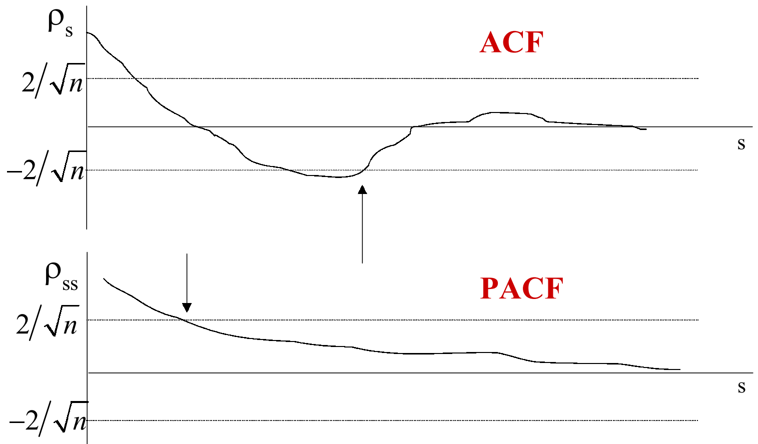
Zero Region $\hat{\rho}_s$ and $\hat{\rho}_{ss} \in 0 \pm \frac{2}{\sqrt{n}}$

With $\rho_s = 0$, $\hat{\rho}_s \sim \mathrm{N}\left(0, \frac{1-(\hat{\rho}_s)^2}{n-2}\right)$

Given $\hat{\rho}_s \approx 0$ and n is large,

95% CI for $\rho_s = 0 \pm 1.96\frac{1}{\sqrt{\mathrm{n}}}$

Box-Jenkins Method (12)



Box-Jenkins Method (13)

<u>Rule</u>

If Y is stationary, try to identify (p,q).

Otherwise, increase order d (d=d+1) or try log(Y) if not conflict with underlying theories. Take difference of Y and repeat correlogram.

Box-Jenkins Method (13)

<u>Identification Rule</u>

If Y~MA(q), #non-zero ACF identifies q

If Y~AR(p), #non-zero PACF identifies p

If Y~ARMA(p,q), ACF and PACF may over-identify (p,q).

Experiment with neighborhood of identified value of (p,q)

Box-Jenkins Method (14)

<u>Estimation ARMA</u>

AR part

- Cochrane-Orcutt Iterative Methods
- Durbin two-step

MA part

- FGLS

Straightforward for EViews. Use LS with lagged Y and MA terms, e.g.,

ls Y Y(-1) Y(-2) MA(1) MA(2)

Box-Jenkins Method (15)

Diagnosis Checking

- check AR roots for stability
- check MA roots for invertibility
- test for Zero Correlation of residuals (make sure that ε_t is White noise)

EViews' features and tricks

- Inverted MA roots
- Inverted AR roots (if used AR terms)
- Box-Pierce and Ljung-Box Q-tests

Box-Jenkins Method (16)

Box-Pierce's Q-test

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_p$$

$$H_1 : \rho_1 \neq \rho_2 \neq \cdots \neq \rho_p$$

$$Q = n\sum_{s=1}^{p} \hat{\rho}_s^2 \sim \chi^2(p)$$

Proof $\sqrt{n}\hat{\rho}_s \sim N(0,1)$ for all s

$$\Longrightarrow Q = n\hat{\rho}_1^2 + n\hat{\rho}_2^2 + \cdots + n\hat{\rho}_p^2 \sim \chi^2(p)$$

Note that Box-Pierce's Q is not OK for small n

Box-Jenkins Method (17)

Ljung-Box's modified Q-test

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_p$$

$$H_1 : \rho_1 \neq \rho_2 \neq \cdots \neq \rho_p$$

$$Q = n(n+2)\sum_{s=1}^{p} \frac{\hat{\rho}_s^2}{n-s} \sim \chi^2(p)$$

Ljung-Box's Q is valid for small n. It has replaced Box-Pierce Q-test