Auto-correlation of Error Terms

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Covered Topics

- GLS
- Assumptions
- Detection, e.g.,
  - Durbin-Watson d-test
  - Breusch-Godfrey test
- Estimation Methods, e.g.,
  - Cochrane-Orcutt

General Auto-correlation (1)

$$\mathbf{Y} = \mathbf{X\beta} + \mathbf{\nu}$$

$$\mathbf{E(\nu) = 0}$$

$$\mathbf{V(\nu)} = \sigma^2 \mathbf{\Sigma}$$

where

0 is a nx1 column vector of zeroes

Σ is an nxn positive-definite symmetric matrix.

General Auto-correlation (2)

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdot & \sigma_{1n} \\ \sigma_{12} & \sigma_{22} & \cdot & \sigma_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{1n} & \sigma_{2n} & \cdot & \sigma_{nn} \end{bmatrix}$$

where $\sigma^2 \sigma_{ij} = V(\nu_i, \nu_j)$ for $i, j = 1, ..., n$

General Auto-correlation (3)

Sources of Auto-correlation

- Inertia (nature)
- Spill-over effect over geographical region, e.g., contagion, migration (nature)
- Spec. errors, e.g.,
  - Exclusion of auto-correlated independent variables
  - Incorrect functional form
  - Lagged terms

General Auto-correlation (4)

Effect of Ignoring Pure Auto-correlation

- OLS is unibiased but not the best
- Need new estimate of Σ, e.g.,
  - Newey-West formula

Generalized LS (1)

$$\mathbf{\Omega Y} = \mathbf{\Omega X \beta} + \mathbf{\Omega \nu}$$

where Ω is an nxn symmetric matrix such that

$$\mathbf{\Omega\Omega} = \mathbf{\Sigma}^{-1}$$

Note that $E(\Omega\nu)=\mathbf{0}$ and $V(\Omega\nu)=\sigma^2\mathbf{I}_n$

Generalized LS (2)

If Σ is known, apply OLS to BLU estimate β,σ^2

$$\hat{\boldsymbol{\beta}} = \left[(\mathbf{\Omega X})^T(\mathbf{\Omega X})\right]^{-1}(\mathbf{\Omega X})^T(\mathbf{\Omega Y})$$

$$= \left[\mathbf{X}^T\mathbf{\Sigma}^{-1}\mathbf{X}\right]^{-1}\mathbf{X}^T\mathbf{\Sigma}^{-1}\mathbf{Y}$$

$$\hat{\sigma}^2 = \tfrac{1}{n-K}\left(\mathbf{\Omega}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})\right)^T\left(\mathbf{\Omega}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})\right)$$

$$= \tfrac{1}{n-K}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})^T\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}})$$

What if Σ is unknown? Need more assumptions?

Assumptions

$$Y_i = X_{1i}\beta_1 + X_{2i}\beta_2 + \ldots + X_{Ki}\beta_K + \nu_i$$
$$i = 1,\ldots,n$$

In addition to CLRM assumptions

- ARMA error term (ν)
- weakly stationary error term. Otherwise, estimation will be invalid. Why?

AR(1) error term --(1)

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \rho\nu_{t-1} + \varepsilon_t$$
$$-1 < \rho < 1, \quad t = 1,\ldots,n$$

$\varepsilon_t \sim$ White Noise

$H_0 : \rho = 0$

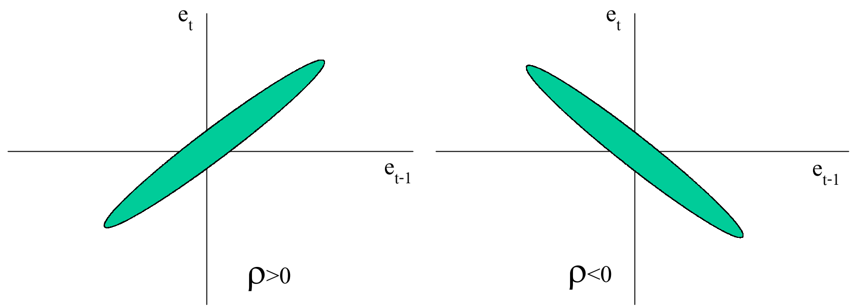
$H_1 : \rho \neq 0$

Accept ==> No auto-correlation

Reject ==> AR(1)

AR(1) error term --(2)

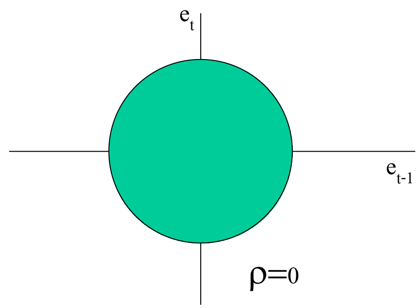
Graphical Test



where ***e*** is the OLS residual when auto-correlation has been ignored

AR(1) error term --(3)

Graphical Test (cont'd)



Note that plot of e_t vs e_{t-1} cannot reveal higher AR in error terms

AR(p) error term --(1)

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \rho\nu_t$$

$$\nu_t = \rho_1\nu_{t-1} + \rho_2\nu_{t-2} + \ldots + \rho_p\nu_{t-p} + \varepsilon_t$$

$\sim$ stationary

$\varepsilon_t \sim$ White Noise

$$H_0 : \rho_1 = \rho_2 = \ldots = \rho_p = 0$$

$$H_1 : \rho_1 \neq \rho_2 \neq \ldots \neq \rho_p \neq 0$$

Accept ==> No auto-correlation

Reject ==> AR(p) or lower

Auto-correlation Tests

Statistical Tests

- Durbin-Watson's d-test. Good only for AR(1).
- DW's h-test (obsolete)
- BG's General Auto-correlation or Serial Correlation test
- Runs test (non-parametric). Not require normal error term nor ARMA form. Good for small sample.

Durbin-Watson's d-test (1)

$$DW = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$$

Note that t=2

DW statistic follows a special distribution. Need a special table to perform AR(1) test.

See Table D.5A-B (Gujarati)

Durbin-Watson's d-test (2)

$$DW = \frac{\sum_{t=2}^{n}(e_t^2 - 2e_{t-1}e_{t-1} + e_{t-1}^2)}{\sum_{t=1}^{n} e_t^2}$$

$$= \frac{\sum_{t=2}^{n} e_t^2}{\sum_{t=1}^{n} e_t^2} - 2\frac{\sum_{t=2}^{n} e_{t-1}e_{t-1}}{\sum_{t=1}^{n} e_t^2} + \frac{\sum_{t=2}^{n} e_{t-1}^2}{\sum_{t=1}^{n} e_t^2}$$

$$\approx 1 - 2\hat{\rho}x + 1 = 2(1 - \hat{\rho})$$

==> $0 \leq DW \leq 4$

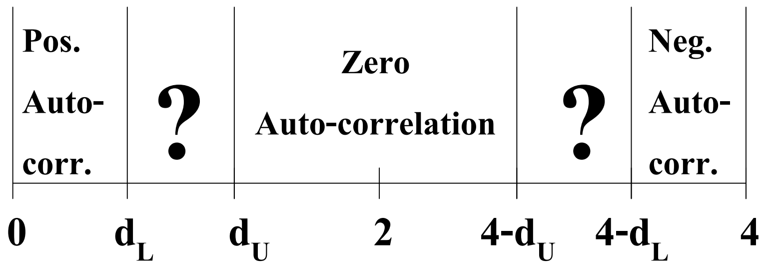
Durbin-Watson's d-test (3)

Given significant level(α) and sample size (n), read d_L and d_U from DW tables.

Criterion:

$$0 \le DW \le d_L \text{ ==> } \rho > 0 \text{ pos. auto-corr.}$$
$$d_L \le DW \le d_U \text{ ==> } \rho \ge 0 \text{ indecisive}$$
$$d_U \le DW \le 4 - d_L \text{ ==> } \rho = 0 \text{ zero auto-corr.}$$
$$4 - d_L \le DW \le 4 - d_U \text{ ==> } \rho \le 0 \text{ indecisive}$$
$$4 - d_U \le DW \le 4 \text{ ==> } \rho < 0 \text{ neg. auto-corr.}$$

Durbin-Watson's d-test (4)



Note that the conclusion is not simply "Accept" or "Reject".

Durbin-Watson's d-test (4)

Notes

- DW is invalid if there are lagged dependent variables as explanatory variables (AR Model). Their existence is equivalent to higher AR of error terms
- Good only for AR(1). DW statistic is usually an item in the OLS report.
- DW can be used to roughly estimate ρ but no SE given.

Breusch-Godfrey's test (1)

Step1 Run OLS by ignoring auto-correlation

==> residual $\hat{v}_t$

Step 2 run OLS for

$$\hat{v}_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \rho_1\hat{v}_{t-1} + \rho_2\hat{v}_{t-2} + \ldots + \rho_p\hat{v}_{t-p} + \varepsilon_t'$$

Breusch-Godfrey's test (2)

Step 3 Do F-Test or χ^2 test for

$$H_0 : \rho_1 = \rho_2 = \ldots = \rho_p = 0$$

$$H_1 : \rho_1 \neq \rho_2 \neq \ldots \neq \rho_p \neq 0$$

Accept H_0 => no AR(p) auto-corr.

Reject ==> AR(p) or lower

Note that BG test also gives estimate for ρ's and their SE's but still not the best estimates.

Breusch-Godfrey's test (3)

$$F_{cal} = \frac{(RSS_R - RSS_U) \div p}{RSS_U \div (n-K-p)}$$

$$= \frac{(TSS_U - RSS_U) \div p}{RSS_U \div (n-K-p)}$$

$$= \frac{n-K-p}{p} R^2 \sim F(p, n-K-p)$$

$$\chi^2_{cal} = (n-p)R^2 \sim \chi^2(p)$$

Cochrane-Orcutt Method (1)

For AR(p) error term

Step1 Run OLS by ignoring auto-correlation

==> residual $\hat{\nu}_t$

Step 2 run OLS for

$$\hat{\nu}_t = \rho_1 \hat{\nu}_{t-1} + \rho_2 \hat{\nu}_{t-2} + \ldots + \rho_p \hat{\nu}_{t-p} + \varepsilon'_t$$

$$\Longrightarrow \hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_p$$

Step 3 Transform ***Y*** and ***X's***

Cochrane-Orcutt Method (2)

Step 3 Transform ***Y*** and ***X's***

$$Y_t^* = Y_t - \hat{\rho}_1 Y_{t-1} - \hat{\rho}_2 Y_{t-2} - \ldots - \hat{\rho}_p Y_{t-p}$$

$$X_{kt}^* = X_{kt} - \hat{\rho}_1 X_{k,t-1} - \hat{\rho}_2 X_{k,t-2} - \ldots - \hat{\rho}_p X_{k,t-p}$$

$$k = 1, .., K$$

Step 4 Run OLS for

$$Y_t^* = X_{1t}^* \beta_1 + X_{2t}^* \beta_2 + \ldots + X_{Kt}^* \beta_K + \nu_t^*$$

$$\nu_t^* \approx \nu_t - \rho_1 \nu_{t-1} - \rho_2 \nu_{t-2} - \ldots - \rho_p \nu_{t-p}$$

$$= \varepsilon_t$$

Cochrane-Orcutt Method (3)

$$V(\nu_t^*) \approx V(\varepsilon_t) = \sigma^2$$

==>OLS is almost BLUE

<u>Step 5</u> Re-calculate $\hat{\nu}_t$ using new $\hat{\boldsymbol{\beta}}$

$$\hat{\nu}_t = Y_t - X_{1t}\hat{\beta}_1 - X_{2t}\hat{\beta}_2 - \ldots - X_{Kt}\hat{\beta}_K$$

If solution does not significantly change, stop. Otherwise, go back to Step 2

MA Part Estimation (1)

Note that

$$\nu_t^* = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \ldots + \theta_q\varepsilon_{t-q}$$

$$Cov(\nu_t^*, \nu_t^*) = \sigma^2(\theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$$

$$Cov(\nu_t^*, \nu_{t-1}^*) = \sigma^2(\theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{q-1}\theta_q)$$

$$\vdots$$

$$Cov(\nu_t^*, \nu_{t-q}^*) = \sigma^2(\theta_1\theta_q)$$

MA Part Estimation (2)

$$\hat{\gamma}_1(\theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{q-1}\theta_q) = (\theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$$

$$\hat{\gamma}_2(\theta_1\theta_3 + \theta_2\theta_4 + \ldots + \theta_{q-2}\theta_q) = (\theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$$

$$\vdots$$

$$\hat{\gamma}_q(\theta_1\theta_q) = (\theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$$

q unknown q equations ==> solve for $\hat{\boldsymbol{\theta}}$

Add as step 4.1 in Cochrane-Orcutt