ARCH Models

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Auto-
Regressive
Conditional
Heteroscedasticity

Covered Topics

- Model Specifications of ARCH
- ARCH tests
- Generalized ARCH or GARCH
- Various Forms of GARCH
- Estimation Method
- Prediciton with GARCH

ARCH Specification

A kind of <u>heteroscedasticity</u> problems.

==> Variance of error terms is not constant over observation. It depends on magnitude of the past errors

<u>Mean equation</u>

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \nu_t$$

$$t = 1, \ldots, n$$

<u>Variance equation</u>

$$\sigma_t^2 = v(\nu_{t-1}^2, \nu_{t-2}^2, \ldots)$$

$$\text{where } \sigma_t^2 = V(\nu_t)$$

Linear ARCH ---(1)

ARCH(1) Error terms

$$\sigma_t^2 = \gamma_0 + \gamma_1 \nu_{t-1}^2$$

ARCH(q) Error terms

$$\sigma_t^2 = \gamma_0 + \gamma_1 \nu_{t-1}^2 + \gamma_2 \nu_{t-2}^2 + \ldots + \gamma_q \nu_{t-q}^2$$

Easy but could yield negative estimated variance

Linear ARCH ---(2)

Re-write Variance equation for ARCH(q)

$$\sigma_t^2 = E(\nu_t^2) - \{E(\nu_t)\}^2$$

Assume that

$$\nu_t^2 = E(\nu_t^2) + \varepsilon_t = \sigma_t^2 + \varepsilon_t$$

where $\varepsilon_t \sim$ white noise

$$\nu_t^2 = \gamma_0 + \gamma_1 \nu_{t-1}^2 + \gamma_2 \nu_{t-2}^2 + \ldots + \gamma_q \nu_{t-q}^2 + \varepsilon_t$$

Exponential ARCH (1)

Exp. ARCH(1) Error terms

$$\ln\sigma_t^2 = f(\nu_{t-1}^2)$$

Examples

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \ln(\nu_{t-1}^2) \quad \text{---(a)}$$

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \left|\frac{\nu_{t-1}}{\sigma_{t-1}}\right| \quad \text{---(b)}$$

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \frac{\nu_{t-1}}{\sigma_{t-1}} \quad \text{---(c)}$$

Note that γ_1 in (a) is <u>unitless</u> elasticity while those in (b) and (c) are rate of change. They are also unitless.

Exponential ARCH (2)

Exp. ARCH(q) Error terms

$$\ln\sigma_t^2 = f(\nu_{t-1}^2, \nu_{t-2}^2, \ldots, \nu_{t-q}^2)$$

Examples

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \ln(\nu_{t-1}^2) + \ldots + \gamma_q \ln(\nu_{t-q}^2) \quad \text{---(a)}$$

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \left|\frac{\nu_{t-1}}{\sigma_{t-1}}\right| + \ldots + \gamma_q \left|\frac{\nu_{t-q}}{\sigma_{t-q}}\right| \quad \text{---(b)}$$

$$\ln\sigma_t^2 = \gamma_0 + \gamma_1 \frac{\nu_{t-1}}{\sigma_{t-1}} + \ldots + \gamma_q \frac{\nu_{t-q}}{\sigma_{t-q}} \quad \text{---(c)}$$

Always yield positive estimated variance but more generally complicated

Exponential ARCH (3)

We can claim that

$$E\left(\frac{\nu_t^2}{\sigma_t^2}\right) = 1, \quad \forall t$$

$$E\left(\ln \frac{\nu_t^2}{\sigma_t^2}\right) \le \ln(1) = 0 \quad \text{according to Jensen's inequality}$$

$$E\left(\ln(\nu_t^2)\right) = \ln(\sigma_t^2) - \delta_t, \quad \delta_t > 0, \forall t$$

$$V\left(\ln(\nu_t^2)\right) = \omega_t > 0, \quad \forall t$$

Assume that δ_t and ω_t are time-invariant

$$\ln(\nu_t^2) = \ln(\sigma_t^2) - \delta + \varepsilon_t$$

where ε_t is white noise

ARCH Test (1)

H_0: q=0 ==> v~ARCH(0)

H_1: q>0

or

$$H_0 : \gamma_1 = \gamma_2 = \ldots = \gamma_q = 0$$

$$H_1 : \gamma_1 \ne \gamma_2 \ne \ldots \ne \gamma_q \ne 0$$

ARCH Test (2)

Linear ARCH(q)

Step 1 Run OLS for mean equation by ignoring ARCH

==> residual or $\hat{\nu}$

Step 2 Run OLS for the variance equation

$$\hat{\nu}_t^2 = \gamma_0 + \gamma_1 \hat{\nu}_{t-1}^2 + \gamma_2 \hat{\nu}_{t-2}^2 + \ldots + \gamma_q \hat{\nu}_{t-q}^2 + \varepsilon_t$$

ARCH Test (3)

Step 3 Do overall F-test or Chi-square (nR^2) test

Accept H_0 ==> No ARCH and estimation in Step 1 is BLUE

Otherwise, need to re-estimate or simply re-calculate coeff. variance-covariance matrix using Newey-West formula

GARCH Specification

Extension of ARCH

==> Variance of error terms also depends on the past variances

<u>Variance equation</u>

$$\sigma_t^2 = v(\nu_{t-1}^2, \nu_{t-2}^2, \ldots, \sigma_{t-1}^2, \sigma_{t-2}^2, \ldots)$$

Linear GARCH ---(1)

GARCH(p,q) Error terms

$$\sigma_t^2 = \gamma_0 + \alpha_1\sigma_{t-1}^2 + \alpha_2\sigma_{t-2}^2 + \ldots + \alpha_p\sigma_{t-p}^2$$

***p* GARCH terms**

$$+\gamma_1\nu_{t-1}^2 + \gamma_2\nu_{t-2}^2 + \ldots + \gamma_q\nu_{t-q}^2$$

***q* ARCH terms**

Linear GARCH ---(2)

Given that $\nu_t^2 = \sigma_t^2 + \varepsilon_t$

Re-write GARCH(p,q) as

$$\begin{aligned}\nu_t^2 - \varepsilon_t = \gamma_0 &+ \alpha_1(\nu_{t-1}^2 - \varepsilon_{t-1}) \\ &+ \alpha_2(\nu_{t-2}^2 - \varepsilon_{t-2}) \\ &+ \ldots + \alpha_p(\nu_{t-p}^2 - \varepsilon_{t-p}) \\ &+ \gamma_1\nu_{t-1}^2 + \gamma_2\nu_{t-2}^2 + \ldots + \gamma_q\nu_{t-q}^2\end{aligned}$$

Linear GARCH ---(3)

Case $p \le q$

$$\begin{aligned}\nu_t^2 = \gamma_0 &+ (\alpha_1 + \gamma_1)\nu_{t-1}^2 \\ &+ (\alpha_2 + \gamma_2)\nu_{t-2}^2 \\ &+ \ldots + (\alpha_p + \gamma_p)\nu_{t-p}^2 \\ &+ \gamma_{p+1}\nu_{t-p-1}^2 + .. + \gamma_q\nu_{t-q}^2 \\ &+ \varepsilon_t + (-\alpha_1)\varepsilon_{t-1} + \ldots + (-\alpha_p)\varepsilon_{t-p}\end{aligned}$$

AR(q)

MA(p)

Linear GARCH ---(4)

Case $p > q$

$$\nu_t^2 = \gamma_0 + (\alpha_1 + \gamma_1)\nu_{t-1}^2 + (\alpha_2 + \gamma_2)\nu_{t-2}^2 + \ldots + (\alpha_q + \gamma_q)\nu_{t-q}^2 + \alpha_{q+1}\nu_{t-q-1}^2 + \ldots + \alpha_p \nu_{t-p}^2$$

$$+ \varepsilon_t + (-\alpha_1)\varepsilon_{t-1} + \ldots + (-\alpha_p)\varepsilon_{t-p}$$

AR(p)

MA(p)

GARCH Test (1)

H_0: p=q=0 ==> v~GARCH(0,0)

H_1: p>0,q>0

or

$$H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_p = \gamma_1 = \gamma_2 = \ldots = \gamma_q = 0$$

$$H_1 : \alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_p \neq \gamma_1 \neq \gamma_2 \neq \ldots \neq \gamma_q \neq 0$$

GARCH Test (2)

<u>Box-Jenkin Method</u>

(Correlogram of Squared Residuals)

With p as order of MA

and q as order of AR

GARCH Test (3)

<u>Full Estimation and Test</u>

- Maximum Likelihood or FGLS
- z-test
- χ^2-test

Other Forms of GARCH

- ARCH-in-mean or ARCH-M
- Asymmetric ARCH or Threshold ARCH or T-ARCH
- Component GARCH
- Asymmetric Component GARCH

ARCH-M

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \delta\sigma_t + \nu_t \quad \text{--(a)}$$

or

$$Y_t = X_{1t}\beta_1 + X_{2t}\beta_2 + \ldots + X_{Kt}\beta_K + \delta\sigma_t^2 + \nu_t \quad \text{--(b)}$$

Expected Y also depends the error variance. An application in Finance. Expected return (Y_t) depends on the risk (σ_t).

Note that it should be called GARCH-in-mean rather than ARCH-in-mean as σ_t is a GARCH term

TARCH (1)

Sign of errors can influence the expected Y. Define

d_{t-1} = 1 if ν_{t-1} <0

= 0, otherwise

GARCH(p,q) with TARCH

$$\sigma_t^2 = \gamma_0 + \alpha_1\sigma_{t-1}^2 + \alpha_2\sigma_{t-2}^2 + \ldots + \alpha_p\sigma_{t-p}^2$$
$$+ \gamma_{1a}\nu_{t-1}^2 + \gamma_{1b}d_{t-1}\nu_{t-1}^2$$
$$+ \gamma_1\nu_{t-2}^2 + \ldots + \gamma_q\nu_{t-q}^2$$

TARCH (2)

Note that the effect of ν_{t-1}^2 on σ_t^2

= γ_{1a} + γ_{1b} when ν_{t-1} <0

= γ_{1a} , otherwise

TARCH can be further generalized to accommodate the effect of ν_{t-1} in more complicated pattern. For example, γ_{1b} can depend on how negative it is. Sign of other lagged ν_t's can be added.

Component GARCH (1)

Note that GARCH(p,q) is equivalent to an ARMA on the squared errors. It must be stationary in the same sense as an ARMA. If it is non-stationary, how can it be handled?

An ARIMA might generate a negative variance. It is not appropriate here.

Answer is Component GARCH

Component GARCH (2)

Define q_t as the long-run component.

Both the short-run or transitory and the long-run components are assumed behave like GARCH as follows:

<u>Transitory Component</u>

$$\sigma_t^2 - q_t = \alpha_1(\sigma_{t-1}^2 - q_{t-1}) + \ldots + \alpha_p(\sigma_{t-p}^2 - q_{t-p}) + \gamma_1(\nu_{t-1}^2 - q_{t-1}) + \ldots + \gamma_q(\nu_{t-q}^2 - q_{t-q})$$

Component GARCH (3)

<u>Long-run Component</u>

$$q_t = \omega + \rho(q_{t-1} - \omega) + \phi(\nu_{t-1}^2 - \sigma_{t-1}^2)$$

That is, the error variance mean-reverts to q_t. If $q_t = \omega$, the error variance mean-reverts to a constant and a time-variant trend.

However, q_t is unobservable.

EGARCH

<u>General Form</u>

$$\ln\sigma_t^2 = \gamma_0 + \alpha_1 \ln(\sigma_{t-1}^2) + \ldots + \alpha_q \ln(\sigma_{t-q}^2) + \gamma_a\left|\frac{\nu_{t-1}}{\sigma_{t-1}}\right| + +\gamma_b\frac{\nu_{t-1}}{\sigma_{t-1}} + \ldots + \gamma_{qa}\left|\frac{\nu_{t-q}}{\sigma_{t-q}}\right| + \gamma_{qb}\frac{\nu_{t-q}}{\sigma_{t-q}}$$

FGLS (1)

Estimation of GARCH(p,q)

Step 0 Set weight = 1 and $\hat{\sigma}_t$=0 for all t

Step 1 Run WLS on the mean equation with the weights and GARCH-M

==> residual $\hat{\nu}_t$

Step 2 Run OLS for the variance equation with *p* GARCH terms and *q* ARCH terms

$$\hat{\nu}_{t-1}^2 = \gamma_0 + \alpha_1\hat{\sigma}_{t-1}^2 + \ldots + \alpha_p\hat{\sigma}_{t-p}^2$$

$$+\gamma_1\hat{\nu}_{t-1}^2 + \gamma_1\hat{\nu}_{t-2}^2 + \ldots + \gamma_q\hat{\nu}_{t-q}^2 + \varepsilon_t$$

FGLS (2)

==> estimated α and γ or $[\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\gamma}}]$

Step 3 Calculate the fitted values of $\hat{\nu}_t^2$ or new series of $\hat{\sigma}_t^2$

$$\hat{\sigma}_t^2 = \hat{\gamma}_0 + \hat{\alpha}_1\hat{\sigma}_{t-1}^2 + \ldots + \hat{\alpha}_p\hat{\sigma}_{t-p}^2$$

$$+\hat{\gamma}_1\hat{\nu}_{t-1}^2 + \hat{\gamma}_2\hat{\nu}_{t-2}^2 + \ldots + \hat{\gamma}_q\hat{\nu}_{t-q}^2$$

It will be used as the estimate GARCH terms in the next OLS runs for the mean equation in Step 1 and the variance equation in Step 2

FGLS (3)

Note that there is no guarantee that it will always be positive

Step 4 Set the inverse of its square root as weight in WLS for the mean equation.

$$w_t = 1\Big/\sqrt{\widehat{\hat{\nu}_t^2}}$$

If solution converges, stop. Otherwise, go back to Step 1 to re-estimate the mean equation.

FGLS (4)

Note that

- an asymmetric ARCH term can be added in the variance equation without any problem
- LS runs in Step 1 and Step 2 could be Non-linear LS if the equations are non-linear in parameters such as in the case of the component ARCH
- similar estimation could also be done for EGARCH

Prediction

Step 1 Use the estimated variance equation to estimate the error variance $\hat{\sigma}^2_{t+f}$

Step 2 Use the estimated mean equation to calculate $\hat{Y}_{t+f}$

Step 3 Calculate the (1-α)100% prediction interval for

$$Y_{t+f} = \hat{Y}_{t+f} \pm t_{.5\alpha}(n-K)\sqrt{\hat{\sigma}_t^2 + \mathbf{X}_{t+f}\mathbf{V}(\hat{\boldsymbol{\beta}})\mathbf{X}^{\mathrm{T}}_{t+f}}$$