# VAR Models

Pongsa Pornchaiwiseskul
Faculty of Economics
Chulalongkorn University



**V**ector

**A**uto-

**R**egressive

Model



# Covered Topics

- What is VAR?
- VAR Estimation
- Granger Causality Test
- Co-integration
- Restricted VAR or VEC
- VEC Estimation



# VAR(p) Specification

A multiple-equation model for more than one endogenous variable

$$\mathbf{Y}_t = (\mathbf{Y}_{1t}, \mathbf{Y}_{2t}, .., \mathbf{Y}_{Mt})^T \text{ for all t}$$

Analogous to AR

$$\mathbf{Y_t} = \Phi_0 + \Phi_1 \mathbf{Y_{t-1}} + \Phi_2 \mathbf{Y_{t-2}} + \ldots + \Phi_p \mathbf{Y}_{\mathbf{t}-p} + \boldsymbol{\varepsilon}_\mathbf{t}$$

where $\Phi_0$ is a Mx1 vector

$\Phi_j$ is an MxM matrix for j=1,...,p

$\boldsymbol{\varepsilon}_\mathbf{t} = (\varepsilon_{1t}, \varepsilon_{2t}, .., \varepsilon_{Mt})^T \sim IID$

# VAR Estimation

Equation-by-equation OLS is valid estimation but not valid for Structural VAR

Structural VAR(p)

$$\mathbf{Y_t} = \Phi_0 + \mathrm{B}\mathbf{Y_t} + \Phi_1 \mathbf{Y_{t-1}} + \Phi_2 \mathbf{Y_{t-2}} + \ldots + \Phi_p \mathbf{Y_{t-p}} + \nu_t$$

where B is an MxM matrix

with zero diagonal elements

$V(\nu_t) = \sigma^2 \Sigma$

Need other estimation methods, e.g., 2SLS, 3SLS, GMM, ML



# Granger Causality (1)

- Granger Causality is a test intended to check for the dynamic relationship between time series variables. It can not reveal the exact AR model
- Some use GC test as a pre-test before a VAR model is set up. However, rejection in GC does not confirm the form of VAR



# Granger Causality (2)

Test Model for Granger-Causality of X on Y

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \ldots + \alpha_p Y_{t-p} + \beta_1 X_{t-1} + \beta_1 X_{t-2} + \ldots + \beta_p X_{t-p} + \varepsilon_t$$

$H_0$: $\beta_1 = \beta_2 = \ldots = \beta_p$

$H_1$: $\beta_1 \neq \beta_2 \neq \ldots \neq \beta_p$

Accept $H_0$ ==> X does not Granger-cause Y

Reject $H_0$ ==> X Granger-causes Y



# Granger Causality (3)

Note:

- Rejection does not imply that X directly cause Y as there are no other variables in the test model.
- The causality of X on Y could come from causality of X on Z and causality of Z on Y
- It is analogous to a correlation and a partial correlation

  Correlation => GC test model

  Partial Corr. => AR model

# Granger Causality (4)

<u>Note</u>

- Similarly, GC test can be performed for causality of Y on X. It is possible that X does not Granger-cause Y but Y Granger-causes X or vice versa.
- In my opinion, GC test is not very useful as its result does not tell much about the form of VAR or AR models. It can be used to determine exogenous variables. However, there are better tests for exogeneity



# Cross-correlogram

- Plot of correlation between $\boldsymbol{Y}_t$ and $\boldsymbol{X}_{t-j}$ against *j*. Note that *j* could be a negative integer
- Similar interpretation as Correlogram in Box-Jenkins method.

  ==> order p
- Akaike Information Criterion (AIC) can be used to determine the order p for VAR but don't put all your bet on it
- See EViews demo



# Co-integration (1)

- VAR estimation is invalid if Y's are not co-integrated at all. That is, there is no stationary linear combination (long-term linear relationship) of Y's. Why? Otherwise, $\varepsilon$ can't be stationary. $\varepsilon$ is equivalent to a linear combination of **Y**'s and their lagged values
- There could be more than one stationary linear combinations but not more than M, of course.



# Co-integration (2)

- VAR can be considered as a short-term model and the set of co-integrating equations (CE's) as the long-term linear model.
- Existence of CE is equivalent to existence of long-term linear model
- Define *m* as the number of CE's or the number of endogenous variables for long term
- *m* can be less than the number of endogenous variables for short term (M).

# Johansen's CI Test (1)

Re-write VAR(p) with intercept and linear time trend as

$$\Delta \mathbf{Y_t} = \Phi_0 + \delta t + \Gamma \Phi_1 \mathbf{Y_{t-1}} + A_1 \Delta \mathbf{Y_{t-1}} + \ldots + A_{p-1} \Delta \mathbf{Y_{t-p+1}} + \boldsymbol{\varepsilon_t}$$

where $\Gamma = \Phi_p + \Phi_{p-1} + \ldots + \Phi_1 - \mathbf{I}$

$$A_1 = -(\Phi_p + \Phi_{p-1} + .. + \Phi_1)$$

$$A_2 = -(\Phi_p + \Phi_{p-1} + .. + \Phi_2)$$

$$\vdots$$

$$A_{p-2} = -(\Phi_p + \Phi_{p-1})$$

$$A_{p-1} = -(\Phi_p)$$



# Johansen's CI Test (2)

Analogy to ADF Test

$H_0$: $\text{Rank}(\Gamma) = M$

$H_1$: $\text{Rank}(\Gamma) < M$

Step 1 Estimate VAR(p)

==> $\hat{\Gamma}$

Step 2 Calculate and sort eigenvalues of $\hat{\Gamma}$

==> $0 < \hat{\lambda}_1 < \hat{\lambda}_1 < \ldots < \hat{\lambda}_M < 1$



# Johansen's CI Test (3)

Note that $\det(\Gamma) = \lambda_1 \lambda_1 \ldots \lambda_M$

$\lambda_1 > 0 ==> \text{Rank}(\Gamma) = M$

LR Test $H_0$: $\lambda_1 = 0$

$H_1$: $\lambda_1 > 0$

$trace = -2\ln(1-\lambda_1) \sim$ Johansen-Juselius $\text{dist}^{\underline{n}}$

where N = sample size

Accept $H_0$ => Rank <M

Set L=1 and go to Step 3

Reject $H_0$ => Full rank. Go to step 5



# Johansen's CI Test (4)

Step 3 Set L=L+1 and test

$H_0$: $\lambda_1 = \lambda_2 = \ldots = \lambda_L = 0$

$H_1$: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_L$

$$trace = -N \prod_{j=1}^{L} \ln(1-\lambda_j)$$

Accept $H_0$ => Rank($\Gamma$) <M-L

repeat Step 3

Reject $H_0$ => Rank ($\Gamma$) =M-L

go to Step 4

# Johansen's CI Test (5)

Step 4

Rank (Γ) = M ==> all endo. Y's are stationary

Rank(Γ) = the number of CE's but it can not say which Y is endo or exog.

Johensen also introduced an exogeneity test.

Endo. Var's can be arbitrarily selected.



# Impluse Response (1)

Response of Y on shocks

$$\frac{\partial Y_{i,t+\tau}}{\partial \varepsilon_{jt}} = ?$$

$$\mathbf{Y_t} = \Phi_0' + \Phi^{-1}(L)\boldsymbol{\varepsilon}_\mathbf{t}$$

where

$$\Phi(L) = \mathbf{I} - \Phi_1 L - \ldots - \Phi_p L^p$$

$$\Phi^{-1}(L) = \mathbf{I} + \Theta_1 L + \ldots + \Theta_\tau L^\tau - \ldots$$



# Impluse Response (2)

Response of Y on shocks

$$\frac{\partial Y_{i,t+\tau}}{\partial \varepsilon_{jt}} = \theta_{\tau ij}$$

where

$$\Theta_\tau = \begin{bmatrix} \theta_{\tau 11} & \theta_{\tau 12} & . & \theta_{\tau 1M} \\ \theta_{\tau 21} & \theta_{\tau 22} & . & \theta_{\tau 2M} \\ . & . & . & . \\ \theta_{\tau M1} & \theta_{\tau M2} & . & \theta_{\tau MM} \end{bmatrix}$$

=>Simultaneous Impulse Response. EViews gives different IR by assuming response ordering.



# Variance Decomposition

Varaince of $\boldsymbol{Y}_{t+\tau}$

Given that certainty at time t+1 and beyond is not yet resolved

$$V(Y_{i,t+\tau}) = \sum_{j=1}^{M} \sum_{s=1}^{\tau} \theta_{sij}^2 V(\varepsilon_{jt+s})$$

==> decompose variance into M components. Each is due to the responsible shock. It implies the source of variations for $Y_t$ in the past τ periods.

# Vector Error Correction

Restricted VAR

analogous to restricted AR

$$\Delta \mathbf{Y_t} = \boldsymbol{\gamma}_0 + \boldsymbol{\delta}\boldsymbol{\varepsilon}'_{\mathbf{t}-1} + \Gamma_1 \Delta \mathbf{Y}_{\mathbf{t}-1} + \ldots + \Gamma_{p-1} \Delta \mathbf{Y}_{\mathbf{t}-p+1} + \boldsymbol{\varepsilon}_\mathbf{t}$$

where

$\boldsymbol{\varepsilon}'_{\mathbf{t}-1}$ is long-term errors in period t-1