ARDL Models

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Auto-

Regressive

Distributed

Lag

Covered Topics

- Generalized Specifications of ARDL
- Selected Forms of DL
  - –Koyck
  - –**Polynomial DL (PDL)**

ARDL Specification (1)

- A form of short-term model
- Explanatory variables
  - lagged values of dependent variable
  - current and lagged values of exogenous variables
- Three components
  - Auto-Regressive part
  - Distributed-Lag part
  - random part

ARDL Specification (2)

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p}$$
$$+ \beta_{10} X_{1t} + \beta_{11} X_{1,t-1} + \ldots + \beta_{1,p1} X_{1,t-p1}$$
$$+ \beta_{20} X_{2t} + \beta_{21} X_{2,t-1} + \ldots + \beta_{2,p2} X_{2,t-p2}$$
$$\vdots$$
$$+ \beta_{K0} X_{Kt} + \beta_{K1} X_{K,t-1} + \ldots + \beta_{K,pK} X_{K,t-pK}$$
$$+ \varepsilon_t$$

ARDL Specification (3)

Effect of current X_k at time t on Y is distributed over *pk* periods.

β_{k0} = Current or <u>short-run</u> effect of X_k on Y

β_{kj} = j-period delayed effect of X_k on Y

β_k = total or <u>long-run</u> effect of X_k on Y

where $\beta_k = \beta_{k0} + \beta_{k1} + \ldots + \beta_{k,pk}$

ARDL Specification (4)

Note that OLS is valid as lagged Y can be uncorrelated with the error term. If the random part is auto-correlated error terms, OLS becomes invalid. No problem with DL models (without AR part).

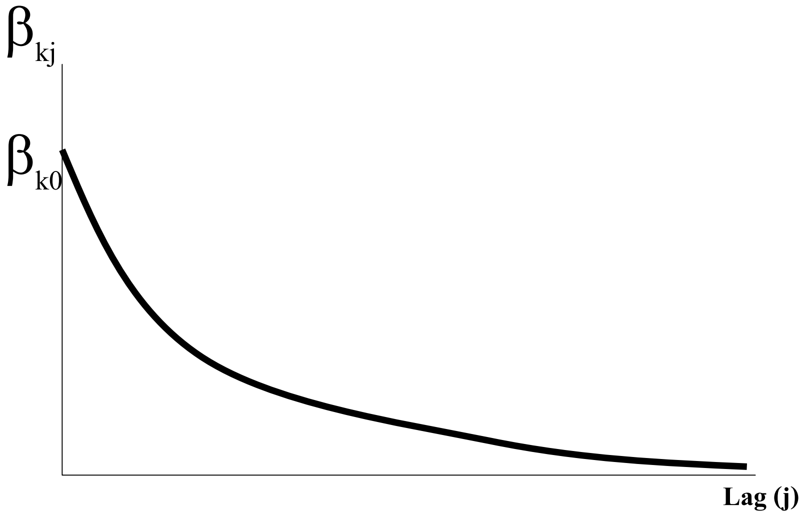
Without restriction on parameters, there will be too many parameters (β), especially for very long lag.

Koyck Models (1)

- No lagged Y on the right-hand side.
- Only one X
- error terms could be ARMA
- Declining effect over infinite lag or
  $pk = \infty$
- Restriction on β_{kj}

$$\beta_j = (\lambda)^j \beta_0\,,\ 0 < \lambda < 1,\quad j = 0,1,\ldots$$

Koyck Models (2)



Koyck Models (3)

$$Y_t = \phi_0 + \beta_0 X_t + \beta_1 X_{t-1} + \ldots + \beta_p X_{t-p} + \varepsilon_t$$

$$= \phi_0 + \beta_0 X_t + \lambda\beta_0 X_{t-1} + \lambda^2 \beta_0 X_{t-2} + \ldots + \varepsilon_t$$

$$= \phi_0 + \beta_0 (1 + \lambda L + \lambda^2 L^2 + \ldots) X_t + \varepsilon_t$$

$$= \phi_0 + \beta_0 (1 - \lambda L)^{-1} X_t + \varepsilon_t$$

$$(1 - \lambda L) Y_t = \phi_0 (1 - \lambda) + \beta_0 X_t + (1 - \lambda L)\varepsilon_t$$

$$Y_t - \lambda Y_{t-1} = \phi_0 (1 - \lambda) + \beta_0 X_t + \varepsilon_t - \lambda\varepsilon_{t-1}$$

$$Y_t = \phi_0 (1 - \lambda) + \lambda Y_{t-1} + \beta_0 X_t + \varepsilon_t - \lambda\varepsilon_{t-1}$$

Koyck Models (4)

Estimation

Step 0 Guess $\hat{\lambda}$

Step 1 Estimate $(\phi_0, \beta_0, \lambda)$ for

$$Y_t - \hat{\lambda} Y_{t-1} = \phi_0 (1 - \lambda) + \beta_0 X_t + \varepsilon_t - \lambda\varepsilon_{t-1}$$

Step 2 Use new $\hat{\lambda}$ and go back to Step 1 until convergence

PDL Models (1)

- No lagged Y on the right-hand side.
- Allow more than one X
- error terms could be ARMA
- polynomial effect over finite lag
- Restriction on β_{kj}

$$\beta_j = \gamma_0 + \gamma_1 j + \gamma_2 j^2 + \ldots + \gamma_m j^m$$

$$m << p, \; j = 0,1,\ldots, p$$

PDL Models (2)

$$Y_t = \phi_0 + \beta_0 X_t + \beta_1 X_{t-1} + \ldots + \beta_p X_{t-p} + \varepsilon_t$$

$$= \phi_0 + \gamma_0 X_t$$

$$+ (\gamma_0 + \gamma_1 + \gamma_2 + \ldots + \gamma_m) X_{t-1}$$

$$+ (\gamma_0 + 2\gamma_1 + 4\gamma_2 + \ldots + 2^m \gamma_m) X_{t-2}$$

$$+ (\gamma_0 + 3\gamma_1 + 9\gamma_2 + \ldots + 3^m \gamma_m) X_{t-3}$$

$$\vdots$$

$$+ (\gamma_0 + p\gamma_1 + p^2\gamma_2 + \ldots + p^m \gamma_m) X_{t-p}$$

$$+ \varepsilon_t$$

PDL Models (3)

$$Y_t = \phi_0$$

$$+ \gamma_0 \left(X_t + X_{t-1} + \ldots + X_{t-p} \right) \quad \leftarrow Z_{0t}$$

$$+ \gamma_1 \left(X_{t-1} + 2X_{t-2} + \ldots + pX_{t-p} \right)$$

$$+ \gamma_2 \left(X_{t-1} + 4X_{t-2} + \ldots + p^2 X_{t-p} \right)$$

$$+ \gamma_3 \left(X_{t-1} + 8X_{t-2} + \ldots + p^3 X_{t-p} \right)$$

$$\vdots$$

$$+ \gamma_m \left(X_{t-1} + 2^m X_{t-2} + \ldots + p^m X_{t-p} \right) \quad \leftarrow Z_{mt}$$

$$+ \varepsilon_t$$

PDL Models (4)

$$Y_t = \phi_0 + \gamma_0 Z_{0t} + \gamma_1 Z_{1t} + \gamma_2 Z_{2t} + \ldots + \gamma_m Z_{mt} + \varepsilon_t$$

==> Unrestricted OLS is BLUE

PDL Models (5)

Case $\beta_{-1} = 0$

Restriction

$$\gamma_0 - \gamma_1 + \gamma_2 + \ldots + (-1)^m \gamma_m = 0$$

Effect starts from zero
==> Restricted LS is BLUE

PDL Models (6)

Case $\beta_{p+1} = 0$

Restriction

$$\gamma_0 + (p+1)\gamma_1 + (p+1)^2\gamma_2 + \ldots + (p+1)^m\gamma_m = 0$$

Effect dies down to zero.
==> Restricted LS is BLUE

PDL Models (7)

Case $\beta_{-1} = 0, \beta_{p+1} = 0$

Restrictions

$$\gamma_0 - \gamma_1 + \gamma_2 + \ldots + (-1)^m\gamma_m = 0$$

$$\gamma_0 + (p+1)\gamma_1 + (p+1)^2\gamma_2 + \ldots + (p+1)^m\gamma_m = 0$$

Effect starts from zero and finally back to zero. ==> Restricted LS is BLUE