Chapter 2

Linear Transformations

2.1 Linear Transformations

Definition 2.1.1. Let V and W be vector spaces over the same field F. A function $T : V \to W$ is said to be a *linear transformation* if

$$\forall\, v, w \in V\ \forall\, \alpha, \beta \in F\quad T(\alpha v + \beta w) = \alpha T(v) + \beta T(w).$$

The set of all linear transformations from V to W is denoted by $\mathcal{L}(V, W)$.

Note 2.1.2. Let V and W be vector spaces over the same field F.

(i) $T : V \to W$ is a linear transformation if and only if

   (a) $\forall\, v, w \in V\quad T(v + w) = T(v) + T(w)$, and

   (b) $\forall\, v \in V\ \forall\, \alpha \in F\quad T(\alpha v) = \alpha T(v)$.

   Furthermore, let $T : V \to W$ be a linear transformation.

(ii) Let $n \in \mathbb{N}$. Then

$$\forall\, v_1, \ldots, v_n \in V\ \forall\, \alpha_1, \ldots, \alpha_n \in F\quad T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n).$$

(iii) $T(0_V) = 0_W$ since

(iv) The graph of a linear transformation from $\mathbb{R}$ into $\mathbb{R}$ is

Example 2.1.3.

(i) Let V and W be vector spaces over the same field. Define $T : V \to W$ by $T(v) = 0$ for all $v \in V$. Then T is a linear transformation, called the *zero transformation*.

(ii) Let V be a vector space. Define $1_V : V \to V$ by $1_V(v) = v$ for all $v \in V$. Then 1_V is a linear transformation, called the *identity transformation*.

(iii) Let F be a field of char $F = 0$ and $A = [a_{ij}] \in M_{mn}(F)$. Define $T_A : F^n \to F^m$ by $T_A\big((\alpha_1, \ldots, \alpha_n)\big) = (\beta_1, \ldots, \beta_m)$, where $\beta_i = \sum_{j=1}^{n} a_{ij}\alpha_j$ for all $\alpha_j \in F$ and $i = 1, \ldots, m$, alternatively, $(T_A(v))^t = A(v)^t$ for all $v \in F^n$. Then T_A is a linear transformation, called the *multiplication by* A.

(iv) Let $V = C^\infty(\mathbb{R}) := \big\{f : \mathbb{R} \to \mathbb{R} \;\big|\; f^{(n)} \text{ exists and is continuous for all } n \in \mathbb{N}\big\}$. Define $D : V \to V$ by $D(f) = f'$ for all $f \in V$. Then D is a linear transformation, called the *differentiation transformation*.

(v) Let $V = \big\{f : \mathbb{R} \to \mathbb{R} \;\big|\; f \text{ is continuous}\big\}$. Define $T : V \to V$ by $T(f)(x) = \int_0^x f(t)\,dt$ for all $f \in V$ and $x \in \mathbb{R}$. Then T is a linear transformation.

Definition 2.1.4. Let $T \in \mathcal{L}(V, W)$. We define the *kernel* of T, denoted by $\ker T$, and the *image* of T, denoted by $\operatorname{im} T$, as follows:

$$\ker T = \big\{v \in V \;\big|\; T(v) = 0\big\} \qquad \text{and} \qquad \operatorname{im} T = \big\{T(v) \;\big|\; v \in V\big\}.$$

Proposition 2.1.5. *Let $T \in \mathcal{L}(V, W)$. Then*

(i) $\ker T \preceq V$;

(ii) $\operatorname{im} T \preceq W$;

(iii) T is injective if and only if $\ker T = \{0\}$;

(iv) T is surjective if and only if $\operatorname{im} T = W$.

Definition 2.1.6. Let $T \in \mathcal{L}(V, W)$. The *nullity* of T, denoted by $\operatorname{null} T$, is the dimension of $\ker T$. The *rank* of T, denoted by $\operatorname{rank} T$, is the dimension of $\operatorname{im} T$.

Example 2.1.7.

(i) Let T be the zero transformation from a vector space V into a vector space W. Then

$$\ker T = \qquad \operatorname{null} T = \qquad \text{and} \qquad \operatorname{im} T = \qquad \operatorname{rank} T = \qquad .$$

(ii) Let 1_V be the identity transformation on vector space V. Then

$$\ker 1_V = \qquad \operatorname{null} 1_V = \qquad \text{and} \qquad \operatorname{im} 1_V = \qquad \operatorname{rank} 1_V = \qquad .$$

(iii) Let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the differentiation transformation. Then

$$\ker D = \qquad \qquad \text{and} \qquad \operatorname{im} D = \qquad \qquad .$$

Theorem 2.1.8. *Let V and W be vector spaces over the same field F. Define an addition on $\mathcal{L}(V, W)$ and a scalar multiplication as follows:*

$$(f+g)(v) = f(v) + g(v) \qquad \text{for all } f, g \in \mathcal{L}(V, W),\ v \in V,$$
$$(\alpha f)(v) = \alpha f(v) \qquad \text{for all } f \in \mathcal{L}(V, W),\ \alpha \in F,\ v \in V.$$

Then $\mathcal{L}(V, W)$ is a vector space over F.

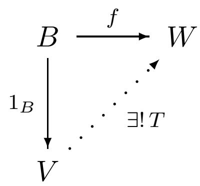
Definition 2.1.9. Let V and W be vector spaces over the same field F and $T : V \to W$ a function. We call T an *(vector space) isomorphism* if

(i) T is a linear transformation; and

(ii) T is a bijection from V onto W.

Moreover, V is *(vector space) isomorphic* to W, denoted by $V \cong W$, if there exists an isomorphism from V onto W.

Theorem 2.1.10. *Let V and W be vector spaces over the same field F, let B be a basis of V and $f : B \to W$ a function. Then there exists a unique linear transformation $T : V \to W$ such that $T(x) = f(x)$ for all $x \in B$.*



This theorem says that we can map the elements of a basis of V to any elements of W we wish, and there will be a **unique** *linear transformation from V into W which has the same action on the basis elements.*

Theorem 2.1.11. *Let $T \in \mathcal{L}(V, W)$ be an isomorphism. Let $S \subseteq V$ and*

$$T(S) = \{T(x) \mid x \in S\}.$$

Then

(i) S spans V if and only if $T(S)$ spans W;

(ii) S is linearly independent in V if and only if $T(S)$ is linearly independent in W;

(iii) S is a basis of V if and only if $T(S)$ is a basis of W.

Theorem 2.1.12. *Let $T \in \mathcal{L}(V, W)$. If B is a basis of V and $T(B)$ is a basis of W, then T is an isomorphism from V onto W.*

Theorem 2.1.13. *Let V and W be vector spaces over the same field. Then*

$$V \cong W \quad \text{if and only if} \quad \dim V = \dim W.$$

Theorem 2.1.14. *Let V and W be finite-dimensional vector spaces over the same field F and $T \in \mathcal{L}(V, W)$. Fix ordered bases $B = \{x_1, \ldots, x_n\}$ of V and $B' = \{y_1, \ldots, y_m\}$ of W. Then*

$$\forall\, j \in \{1, \ldots, n\}\ \exists!\ a_{1j}, \ldots, a_{mj} \in F \quad T(x_j) = \sum_{i=1}^{m} a_{ij} y_i.$$

Moreover, for each $v \in V$, if there exist $\alpha_1, \ldots, \alpha_n \in F$ such that $v = \alpha_1 x_1 + \cdots + \alpha_n x_n$, then

$$T(v) = \beta_1 y_1 + \cdots + \beta_m y_m, \qquad \text{where } \beta_i = \sum_{j=1}^{n} a_{ij} \alpha_j.$$

Definition 2.1.15. The matrix $A = [a_{ij}]$ defined in Theorem 2.1.14 is called the *matrix of T with respect to the (ordered) bases B and B'*, denoted by $m_{B,B'}(T)$.

In the special case where $V = W$ and $B = B'$, we usually just call A the *matrix of T with respect to B*.

Note that Theorem 2.1.14 says that once we have $B = \{x_1, \dots, x_n\}$, $B' = \{y_1, \dots, y_m\}$ and $m_{B,B'}(T)$, then we can calculate $T(v)$ for any $v \in V$. In particular,

$$T(x_j) = \sum_{i=1}^{m} a_{ij} y_i = a_{1j} y_1 + \cdots + a_{mj} y_m,$$

where $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is the jth column vector of A. Moreover,

$$m_{B,B'}(T) = \begin{bmatrix} | & & | \\ T(x_1) & \cdots & T(x_n) \\ | & & | \end{bmatrix}_{m \times n}$$

Theorem 2.1.16. *Let V be an n-dimensional and W an m-dimensional vector spaces over the same field F. Then $\mathcal{L}(V, W) \cong M_{mn}(F)$ and $\dim \mathcal{L}(V, W) = mn$.*

Theorem 2.1.16 says that linear transformations on finite-dimensional vector spaces and matrices are essentially the same mathematically.

Theorem 2.1.17. *Let U, V, W be finite-dimensional vector spaces, $f \in \mathcal{L}(U, V)$, $g \in \mathcal{L}(V, W)$. Moreover, let B, C and D be be ordered bases for U, V and W, respectively. Then $g \circ f \in \mathcal{L}(U, W)$ and*

$$m_{B,D}(g \circ f) = m_{C,D}(g) m_{B,C}(f).$$

Theorem 2.1.18. *Let $A \in M_{kl}(F)$, $B \in M_{mn}(F)$ and $C \in M_{pq}(F)$. Then $(AB)C$ is defined if and only if $A(BC)$ is defined. Moreover, when both are defined, $(AB)C = A(BC)$.*

Definition 2.1.19. Let $A \in M_{nn}(F)$. We say that A is *left invertible* if there exists $B \in M_{nn}(F)$ such that $BA = I_n$, and any matrix B such that $BA = I_n$ is called a *left inverse* of A. Likewise, we call A is *right invertible* if there exists $C \in M_{nn}(F)$ such that $AC = I_n$, and any matrix C such that $AC = I_n$ is called a *right inverse* of A. Finally, we say that A is *invertible* if there exists $D \in M_{nn}(F)$ such that $AD = I_n = DA$, and any matrix D such that $AD = I_n = DA$ is called an *inverse* of A.

Proposition 2.1.20. *Let V be a finite-dimensional vector space, B an ordered basis of V and $T \in \mathcal{L}(V, V)$ a bijection. Then $m_B(T)$ is invertible. Moreover, $m_B(1_V) = [\delta_{ij}] = I_{\dim V}$.*

Proposition 2.1.21. *Let V be an n-dimensional vector space, B an ordered basis of V and $T \in \mathcal{L}(V, V)$.*

(i) T is a bijection if and only if $m_B(T)$ is left invertible.

(ii) T is a bijection if and only if $m_B(T)$ is right invertible.

Theorem 2.1.22. *Let $A \in M_{nn}(F)$. If A is left invertible or A is right invertible, then A is invertible. Moreover, if A is invertible, then A has a unique inverse, and in fact every left inverse and every right inverse of A is an inverse of A.*

Theorem 2.1.23. *Let V and W be finite-dimensional vector space with ordered bases B and B' for V and ordered bases C and C' bases for W. Then for any $T \in \mathcal{L}(V, W)$,*

$$m_{B',C'}(T) = m_{C,C'}(1_W)m_{B,C}(T)m_{B',B}(1_V).$$

In the special case where $V = W$, $B = C$ and $B' = C'$, we obtain that

$$m_{B'}(T) = A^{-1}m_B(T)A, \quad \text{where } A = m_{B',B}(1_V).$$

Corollary 2.1.24. *Let V be a finite-dimensional vector space with ordered bases B and B' for V. Then $m_{B,B'}(1_V)$ is invertible and $\Big(m_{B,B'}(1_V)\Big)^{-1} = m_{B',B}(1_V)$.*

Definition 2.1.25. Let $A = [a_{ij}] \in M_{nn}(F)$. We define the *trace* of A, denoted by $\operatorname{tr} A$, to be the scalar

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

Lemma 2.1.26. *Let V be a finite-dimensional vector space with ordered bases B and C for V and $T \in \mathcal{L}(V, V)$. Then*

$$\operatorname{tr}\Big(m_B(T)\Big) = \operatorname{tr}\Big(m_C(T)\Big).$$

Definition 2.1.27. Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, V)$. Then we define the *trace* of T, denoted by $\operatorname{tr} T$, to be the scalar

$$\operatorname{tr} T = \operatorname{tr} m_B(T),$$

where B is any basis of V.

Proposition 2.1.28. *Let V be an n-dimensional vector space over field F.*

(i) The trace maps $\operatorname{tr} : M_{nn}(F) \to F$ and $\operatorname{tr} : \mathcal{L}(V, V) \to F$ are both linear transformations.

(ii) If $A, B \in M_{nn}(F)$, then $\operatorname{tr} AB = \operatorname{tr} BA$.

Definition 2.1.29. Let V be an n-dimensional vector space, $B = \{v_1, \ldots, v_n\}$ an ordered basis of V. For each $v \in V$, there is a unique ordered n-tuple $(\alpha_1, \ldots, \alpha_n)$ of scalars for which

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

This allows us to associate to each vector $v \in V$ a unique column matrix of length n as follows

$$v \longmapsto [v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

The matrix $[v]_B$ is called the *co-ordinate matrix of v with respect to the ordered basis B*.

Proposition 2.1.30. *Let V be a finite-dimensional vector space and B an ordered basis of V. Then*

$$\forall\, u, v \in V \quad [u+v]_B = [u]_B + [v]_B$$
$$\forall\, \alpha \in F\ \forall\, v \in V \quad [\alpha v]_B = \alpha [v]_B$$

Proposition 2.1.31. *Let V be an n-dimensional vector space over a field F and B an ordered basis of V. Define $\phi_B : V \to F^n$ by $\phi_B(v) = [v]_B^t$ for all $v \in V$. Then ϕ_B is an isomorphism.*

Corollary 2.1.32. *If V is an n-dimensional vector space over a field F, then $V \cong F^n$.*

Definition 2.1.33. Let V be a finite-dimensional vector space over a field F and B and C be ordered bases of V. The *change of basis matrix from B to C*, denoted by $M_{B,C}$, is defined as follows:

$$M_{B,C} = \begin{bmatrix} | & & | \\ [b_1]_C & \cdots & [b_n]_C \\ | & & | \end{bmatrix}_{n \times n} \in M_{nn}(F).$$

Theorem 2.1.34. *Let B and C be ordered bases of a vector space V. Then*

$$[v]_C = M_{B,C}[v]_B \quad \text{for all } v \in V.$$

Theorem 2.1.35. *Let B and C be ordered bases of a vector space V. Then*

$$M_{B,C} = m_{B,C}(1_V).$$

Definition 2.1.36. Let $A \in M_{mn}(F)$. Define $T_A : F^n \to F^m$ by

$$T_A(v) = Av^t \quad \text{for all } v \in F^n.$$

We call T_A the *multiplication by A*.

Proposition 2.1.37.

(i) If $A \in M_{mn}(F)$, then $T_A \in \mathcal{L}(F^n, F^m)$.

(ii) For each $T \in \mathcal{L}(F^n, F^m)$ there exists a unique $A \in M_{mn}(F)$ such that $T = T_A$. This matrix is called the standard matrix for T. *Moreover, the ith column of A is $\big[T(e_i)\big]_C$, where e_i is the standard basis element of the standard basis for F^n and C is the standard basis of F^m, so that*

$$A = \begin{bmatrix} | & & | \\ \big[T(e_1)\big]_C & \cdots & \big[T(e_n)\big]_C \\ | & & | \end{bmatrix}$$

Example 2.1.38. Let $T : F^3 \to F^3$ be defined by $T(x,y,z) = (x-2y, z, x+y+z)$. Then, in column form, [make sure that T is a linear transformation.]

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

Proposition 2.1.39. *For each invertible matrix A and any ordered basis B of a vector space V there exists a unique ordered basis C of V such that $A = M_{B,C}$.*

Theorem 2.1.40. *Let $T \in \mathcal{L}(V,W)$, B and C be ordered bases of V and W, respectively, with $\dim V = n$ and $\dim W = m$. Then T can be represented by $T_A \in \mathcal{L}(F^n, F^m)$, that is*

$$\Big[T(v)\Big]_C = T_A\Big([v]_B\Big),$$

where $A = m_{B,C}(T)$, i.e.,

$$\Big[T(v)\Big]_C = m_{B,C}(T)[v]_B.$$

Moreover when $V = W$ and $B = C$,

$$\Big[T(v)\Big]_B = m_B(T)[v]_B.$$

Note 2.1.41. Let $T \in \mathcal{L}(V,W)$, B and C be ordered bases of V and W, respectively, with $\dim V = n$ and $\dim W = m$. Then, in fact,

$$m_{B,C}(T) = \begin{bmatrix} | & & | \\ \big[T(b_1)\big]_C & \cdots & \big[T(b_n)\big]_C \\ | & & | \end{bmatrix}$$

Example 2.1.42. Let $D : P_2 \to P_2$ be the derivative operator. Let $B = C = \{1, x, x^2\}$. Then

$m_B(D) =$

Moreover, if $p(x) = 5 + x + 2x^2$, then find $D\big(p(x)\big)$.

Definition 2.1.43. The matrices A and B in $M_{mn}(F)$ are *equivalent* if there exist invertible matrices P and Q for which $B = PAQ^{-1}$.

Theorem 2.1.44. *Let $A, B \in M_{mn}(F)$. Then the followings are equivalent.*

(i) If $T \in \mathcal{L}(V,W)$ and C and D are ordered bases of V and W, respectively, and $A = m_{C,D}(T)$, then there exist ordered bases C' and D' of V and W, respectively, such that $B = m_{C',D'}(T)$.

(ii) A and B are equivalent.

Definition 2.1.45. The matrices A and B in $M_{nn}(F)$ are *similar* if there exists an invertible matrix P for which $B = PAP^{-1}$.

Theorem 2.1.46. *Similarity of matrices is an equivalence relation on $M_{nn}(F)$.*

Theorem 2.1.47. *Let $A, B \in M_{nn}(F)$. Then the followings are equivalent.*

(i) If $T \in \mathcal{L}(V,V)$ and C is an ordered basis of V and $A = m_C(T)$, then there exists an ordered basis C' of V such that $B = m_{C'}(T)$.

(ii) A and B are similar.