Chapter 3

Quotient Spaces, Direct Sums and Projections

3.1 Quotient Spaces

Definition 3.1.1. Let V be a vector space and W a subspace of V. For each $v \in V$ the set $v + W = \{v + w \mid w \in W\}$ is called a *coset* of W in V or an *affine subspace* of V. The set of all cosets, i.e., $\{v + W \mid v \in V\}$ is denoted by V/W (read 'V mod W') and is called the *quotient space* of V by W.

One thing we have to be aware of is that it is possible to have $u + W = v + W$ for $u \neq v$, and it certainly seems possible that we could have

$$(u + W) \cap (v + W) \neq \emptyset \quad \text{for some } u \neq v.$$

Lemma 3.1.2. *Let V be a vector space and W a subspace of V. Then, for each $u, v \in V$, either $u + W = v + W$ or $(u + W) \cap (v + W) = \emptyset$. Also, $u + W = v + W$ if and only if $u - v \in W$.*

Theorem 3.1.3. *Let V be a vector space over a field F and W a subspace of V. The quotient space V/W becomes a vector space over F if we let $0 + W$ be the zero vector and define addition and scalar multiplication by*

$$(u + W) + (v + W) = (u + v) + W, \quad \text{and}$$
$$\alpha(v + W) = \alpha v + W \quad \text{for all } u, v \in V \text{ and } \alpha \in F.$$

Theorem 3.1.4. *Let V be a vector space over a field F and W a subspace of V. The map $\pi : V \to V/W$ defined by*

$$\pi(v) = v + W \qquad \text{for all } v \in V,$$

is a surjective linear transformation with $\ker \pi = W$.

Moreover, given any linear transformation $T : V \to V'$ such that $W \subseteq \ker T$, there exists a unique linear transformation $\phi : V/W \to V'$ such that $T = \phi \circ \pi$ and $\ker \phi = (\ker T)/W$.

Definition 3.1.5. The map $\pi : V \to V/W$ defined in Theorem 3.1.4 is called the *canonical map* (or *canonical projection* or *natural projection*) from V onto V/W.

Remark here that the first part of Theorem 3.1.4 says that

every subspace of a vector space is the kernel of some linear transformation (in fact, it is the canonical map).

Corollary 3.1.6. The First Isomorphism Theorem

Let $T \in \mathcal{L}(V, V')$. *Then there exists an isomorphism* $\phi : V/\ker T \to \operatorname{im} T$. *Alternatively,* $V/\ker T \cong \operatorname{im} T$, *i.e.,* $V/\ker T$ *is* isomorphic *to* $\operatorname{im} T$.

The First Isomorphism Theorem says that the image of any linear transformation with domain V is isomorphic to a quotient space of V. Thus, by identifying isomorphic spaces as being essentially the same, we can say that the images of linear transformations on V are just the quotient space of V.

Conversely, any quotient space V/W of V is the image of a linear transformation on V, in particular, V/W is the image of the surjective canonical map $\pi : V \to V/W$. Thus, up to isomorphism, images of linear transformations on V are the same as quotient spaces of V.

Proposition 3.1.7. *Let* W *be a subspace of a vector space* V *and* B, C *be bases of* V *and* W, *respectively, such that* $C \subseteq B$. *Then* $\{v + W \mid v \in B \backslash C\}$ *is a basis of* V/W.

Moreover, if $u, v \in B \backslash C$ *with* $u \neq v$, *then* $u + W \neq v + W$.

Corollary 3.1.8. *Let* W *be a subspace of a vector space* V. *If any two of the vector spaces* V, W *and* V/W *are finite-dimensional, then all three spaces are finite-dimensional and*

$$\dim V/W = \dim V - \dim W.$$

Recall from Definition 2.1.6 that for each $T \in \mathcal{L}(V, V')$, the rank of T, denoted by $\operatorname{rank} T$, is $\dim \operatorname{im} T$ and the nullity of T, denoted by $\operatorname{null} T$, is $\dim \ker T$.

Theorem 3.1.9. The Rank Plus Nullity Theorem

Let $T \in \mathcal{L}(V, V')$ *and assume that* V *is finite-dimensional. Then* $\ker T$ *and* $\operatorname{im} T$ *are also finite-dimensional and*

$$\dim V = \operatorname{rank} T + \operatorname{null} T.$$

3.2 Direct Sums

We proved that for subspaces U and W of a vector space V, the set

$$U + W = \{u + w \mid u \in U \text{ and } w \in W\}$$

is a subspace of V, and in fact $U + W = \langle U \cup W \rangle$. This can easily be extended to more than two subspaces.

Theorem 3.2.1. *If* $W_1, \dots, W_n$ *are subspaces of a vector space* V *and we let*

$$W_1 + \cdots + W_n := \{w_1 + \cdots + w_n \mid w_i \in W_i \text{ for all } i = 1, \dots, n\},$$

then $W_1 + \cdots + W_n$ *is a subspace of* V *and*

$$W_1 + \cdots + W_n = \langle W_1 \cup \cdots \cup W_n \rangle.$$

Note 3.2.2. If $W_1, \ldots, W_n$ are subspaces of a vector space V such that $V = W_1 + \cdots + W_n$, then

$$\forall\, v \in V\ \exists\, w_i \in W_i\ (i = 1, \ldots, n) \quad v = w_1 + \cdots + w_n.$$

In general, the w_i's are not unique.

Definition 3.2.3. Let $W_1, \ldots, W_n$ be subspaces of a vector space V. We say that V is the *direct sum* (or *internal direct sum*) of $W_1, \ldots, W_n$ if

(i) $V = W_1 + \cdots + W_n$, and

(ii) $\forall\, i \in \{1, \ldots, n\} \quad W_i \cap \big(W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_n\big) = \{0\}$.

We denote this by writing $V = W_1 \oplus \cdots \oplus W_n$ or $V = \bigoplus_{i=1}^{n} W_i$.

Moreover, sometimes, we write $W_i \cap \big(W_1 + \cdots + \widehat{W_i} + \cdots + W_n\big)$ instead of $W_i \cap \big(W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_n\big)$.

Theorem 3.2.4. *Let $W_1, \ldots, W_n$ be subspaces of a vector space V such that $V = W_1 + \cdots + W_n$. Then the followings are equivalent.*

(i) $V = W_1 \oplus + \cdots \oplus W_n$.

(ii) If $v_1 + \cdots + v_n = w_1 + \cdots + w_n$, where $v_i, w_i \in W_i$ for all $i \in \{1, \ldots, n\}$, then

$$v_i = w_i \quad \text{for all } i \in \{1, \ldots, n\}.$$

That is, each element of V has a unique representation of the form $w_1 + \cdots + w_n$, where $w_i \in W_i$ for all i.

(iii) If $0 = w_1 + \cdots + w_n$, where $w_i \in W_i$ for all i, then $w_i = 0$ for all i.

*[*Remark: *Some books use this theorem as the definition of* **direct sum***.]*

Proposition 3.2.5. *Let $W_1, \ldots, W_n$ be subspaces of a vector space V such that $V = W_1 \oplus \cdots \oplus W_n$ and suppose that for each $i \in \{1, \ldots, n\}$, S_i is a linearly independent subset of W_i. Then*

$$S_1 \cup \ldots \cup S_n \text{ is linearly independent} \qquad \text{and} \quad \forall\, i, j \in \{1, \ldots, n\} \quad i \neq j \Longrightarrow S_i \cap S_j = \emptyset.$$

Theorem 3.2.6. *Let V be a vector space. For any subspace W of V, there exists a subspace U of V such that $V = U \oplus W$.*

Is the subspace U in Theorem 3.2.6 unique?

Definition 3.2.7. The subspace U of V in Theorem 3.2.6 is called a *complement* of W in V and is denoted by W^c.

Note 3.2.8. Theorem 3.2.6 can be rewritten as

Any subspace of a vector space has a complement.

Theorem 3.2.9. *Let V, V' be vector spaces over the same field and $W_1, \dots, W_n \preceq V$ be such that $V = W_1 \oplus \cdots \oplus W_n$. Suppose that $T_i \in \mathcal{L}(W_i, V')$ for all $i \in \{1, \dots, n\}$. Then there exists a unique linear transformation $T : V \to V'$ such that $T\,\big|_{W_i} = T_i$ for all $i \in \{1, \dots, n\}$.*

Recall that a direct sum is an **internal direct sum**. We can also extend a direct sum in which we must go outside of the given vector spaces to form a new vector space.

Theorem 3.2.10. *Let $V_1, \dots, V_n$ be vector spaces over the same field F.*

(i) The Cartesian product *$V_1 \times \cdots \times V_n$ becomes a vector space over F if we use $(0, \dots, 0)$ as its zero vector, and define addition and scalar multiplication by*

$$(u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n), \text{ and}$$
$$\alpha(v_1, \dots, v_n) = (\alpha v_1, \dots, \alpha v_n),$$

for all $\alpha \in F$ and $(u_1, \dots, u_n), (v_1, \dots, v_n) \in V_1 \times \cdots \times V_n$.

(ii) If, for each $i \in \{1, \dots, n\}$, we let

$$W_i = \big\{(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n \;\big|\; v_j = 0 \text{ for all } j \neq i\big\},$$
$$= \big\{(0, \dots, 0, v, 0, \dots, 0) \in V_1 \times \cdots \times V_n \;\big|\; v \in V_i\big\},$$

then each W_i is a subspace of $V_1 \times \cdots \times V_n$, $W_i \cong V_i$ and

$$V_1 \times \cdots \times V_n = W_1 \oplus \cdots \oplus W_n.$$

Definition 3.2.11. Let $V_1, \dots, V_n$ be vector spaces over the same field. Then $V_1 \times \cdots \times V_n$ defined as in Theorem 3.2.10 is called the *external direct sum of* $V_1, \dots, V_n$, denoted by $V_1 \boxplus \cdots \boxplus V_n$.

3.3 Projections

Definition 3.3.1. Let V be a vector space. A linear transformation $P : V \to V$ is called a *projection* if $P \circ P = P$.

Lemma 3.3.2. *Let V be a vector space and $P : V \to V$ a projection. Then $Q = 1_V - P$ is also a projection and $Q \circ P = P \circ Q = 0$.*

Definition 3.3.3. Let V be a vector space. Two linear transformations $P, Q : V \to V$ are called *supplementary* if $P + Q = 1_V$ and $P \circ Q = 0$.

Lemma 3.3.4. *Let V be a vector space and P, Q be supplementary linear transformations on V. Then P and Q are both projections.*

Proposition 3.3.5. *Let V be a vector space and P a projection on V. Then*

$$V = \ker P \oplus \operatorname{im} P \qquad \text{and} \qquad P\,\big|_{\operatorname{im} P} = 1_{\operatorname{im} P}.$$

From this we can easily see how to prove that, given any subspace W of a vector space V, there is a projection of V onto W.

Proposition 3.3.6. *Let V be a vector space and $W \preceq V$. Then there exists a projection of V onto W, i.e., there exists a projection $P : V \to V$ such that $\operatorname{im} P = W$.*