4.3 Annihilators

Definition 4.3.1. Let M be a non-empty subset of a vector space V. The *annihilator* M^0 of M is

$$M^0 = \{\phi \in V^* \mid \phi(M) = \{0\}\} = \{\phi \in V^* \mid \phi(m) = 0 \text{ for all } m \in M\}.$$

Proposition 4.3.2. *Let M be a non-empty subset of a vector space V. Then M^0 is a subspace of V^*. [Note that M does not need to be a subspace of V.]*

Remark 4.3.3. Let V be a vector space. Then

$$\{0\}^0 = \qquad\qquad \text{and} \qquad V^0 =$$

Theorem 4.3.4. *If W is a subspace of a finite-dimensional vector space V, then*

$$\dim W^0 = \dim V - \dim W.$$

Theorem 4.3.5. *Let V be a vector space.*

(i) For any non-empty subsets M and N of V,

$$M \subseteq N \Longrightarrow N^0 \subseteq M^0.$$

(ii) If V is finite-dimensional, then, identifying V^{**} with V under the natural map, we obtain that

$$M^{00} = \langle M \rangle \quad \text{for any non-empty subset } M \text{ of } V.$$

In particular, if W is a subspace of V, then $W^{00} = W$.

(iii) If V is finite-dimensional and $U, W \preceq V$, then

$$(U \cap W)^0 = U^0 + W^0 \qquad \text{and} \qquad (U + W)^0 = U^0 \cap W^0.$$

Corollary 4.3.6. *Let V be a finite-dimensional vector space and W_1, W_2 subspaces of V. Then*

$$W_1 = W_2 \Longleftrightarrow W_1^0 = W_2^0.$$

Theorem 4.3.7. *Let V be a finite-dimensional vector space and U, W subspaces of V such that $V = U \oplus W$. Then*

(i) $U^* \cong W^0$ [what annihilate W are all linear functionals on U] and $W^* \cong U^0$.

(ii) $(U \oplus W)^* = U^0 \oplus W^0$.

4.4 Operator Adjoints

Definition 4.4.1. Let V and W be vector spaces over the same field F and $\tau \in \mathcal{L}(V, W)$. Define a map $\tau^{\times} : W^* \to V^*$ by

$$\tau^{\times}(\phi) = \phi \circ \tau := \phi\tau \quad \text{for all } \phi \in W^*.$$

[This makes sense, since $\tau \in \mathcal{L}(V, W)$ and $\phi \in \mathcal{L}(W, F)$, we have $\phi\tau \in \mathcal{L}(V, F) = V^*$.]

Moreover, $\tau^{\times}(\phi)(v) = \phi\big(\tau(v)\big)$ for all $\phi \in W^*$ and $v \in V$.

The map $\tau^{\times}$ is called the *operator adjoint* of τ.

Theorem 4.4.2. *Let V and W be vector spaces over the same field F. Then the operator adjoint of $\tau \in \mathcal{L}(V, W)$ is a linear transformation.*

Theorem 4.4.3. *Let V and W be vector spaces over the same field F.*

(i) $\big(\tau + \sigma\big)^{\times} = \tau^{\times} + \sigma^{\times}$ *for all* $\tau, \sigma \in \mathcal{L}(V, W)$.

(ii) $\big(\alpha\tau\big)^{\times} = \alpha\tau^{\times}$ *for all* $\alpha \in F$ *and* $\tau \in \mathcal{L}(V, W)$.

(iii) $\big(\tau\sigma\big)^{\times} = \sigma^{\times}\tau^{\times}$ *for all* $\sigma \in \mathcal{L}(V, W)$ *and* $\tau \in \mathcal{L}(W, U)$.

(iv) $\big(\tau^{-1}\big)^{\times} = \big(\tau^{\times}\big)^{-1}$ *for all invertible* $\tau \in \mathcal{L}(V)$.

Theorem 4.4.4. *Let V and W be finite-dimensional vector spaces over the same field and let $\tau \in \mathcal{L}(V, W)$. If we identify V^{**} with V and W^{**} with W, using the natural maps, then*

$$\tau^{\times\times} = \tau.$$

Theorem 4.4.5. *Let $\tau \in \mathcal{L}(V, W)$. Then*

(i) $\ker \tau^{\times} = \big(\mathrm{im}\,\tau\big)^0$;

(ii) $\big(\mathrm{im}\,\tau^{\times}\big)^0 = \ker \tau$, *under the natural identification with V and W being finite-dimensional;*

(iii) $\mathrm{im}\big(\tau^{\times}\big) \subseteq \big(\ker \tau\big)^0$;

(iv) if V and W are finite-dimensional, then $\mathrm{im}\big(\tau^{\times}\big) = \big(\ker \tau\big)^0$.

Corollary 4.4.6. *Let $\tau \in \mathcal{L}(V, W)$, where V, W are finite-dimensional vector spaces. Then* $\mathrm{rank}\,\tau = \mathrm{rank}\,\tau^{\times}$.