Chapter 4

Dual Spaces

4.1 Dual Spaces

Let F be a field and V a vector space over F. In this chapter, we study linear transformations from V to F by thinking of F as a vector space over itself.

Definition 4.1.1. Let V be a vector space over a field F. The *dual space* of V, denoted by V^*, is the set of all linear transformations from V to F. Thus

$$V^* = \mathcal{L}(V, F) = \{\phi : V \to F \mid \phi \text{ is a linear transformation}\}.$$

The elements of V^* are called *linear functionals on* V.

Note that $\mathcal{L}(V, W)$ is also a vector space whenever V and W are vector spaces. Thus, V^* is always a vector space whenever V is. Hence, what is a basis of V^*?

Proposition 4.1.2. *Let $V \neq \{0\}$ be a finite-dimensional vector space over a field F and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. For each $i \in \{1, \ldots, n\}$, define $\phi_i \in V^*$ by*

$$\phi_i(v_j) = \delta_{ij} \quad \text{for all } j \in \{1, \ldots, n\}.$$

Then $\{\phi_1, \ldots, \phi_n\}$ is a basis of V^*.

In fact, for each i, we define $\psi_i : B \to F$ by $\psi_i(v_j) = \delta_{ij}$ for all j, then there exists a unique linear transformation $\phi_i : V \to F$, i.e., $\phi_i \in V^*$, such that $\phi_i \big|_B = \psi_i$ and this defines a unique element of V^*.

Definition 4.1.3. The basis $\{\phi_1, \ldots, \phi_n\}$ defined in Proposition 4.1.2 is called the *dual basis* of the basis $\{v_1, \ldots, v_n\}$.

Remark 4.1.4. Let V be a vector space. Then

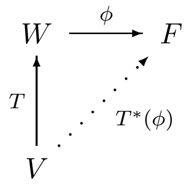
$$\begin{aligned} \forall\ \phi \in V^* \quad & \phi = \phi(v_1)\phi_1 + \cdots + \phi(v_n)\phi_n \\ \forall\ v \in V \quad & v = \phi_1(v)v_1 + \cdots + \phi_n(v)v_n. \end{aligned}$$

Let $v \in V$. Then $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ so that $\phi_i(v) = \alpha_1 \phi_i(v_1) + \cdots + \alpha_n \phi_i(v_n) = \alpha_i$.

Corollary 4.1.5. *Let V be a finite-dimensional vector space. Then $\dim V = \dim V^*$ and $V \cong V^*$.*

Proposition 4.1.6. *Let V, W be vector spaces over a field F and $T \in \mathcal{L}(V,W)$. Define $T^* : W^* \to F^V = \{f \mid f : V \to F\}$ by*

$$T^*(\phi) = \phi \circ T \quad \text{for all } \phi \in W^*, \qquad \text{i.e.,}$$



Then

(i) $T^* \in \mathcal{L}(W^*, V^*)$,

(ii) T^* *is injective if and only if* T *is surjective,*

(iii) T^* *is surjective if and only if* T *is injective.*

4.2 Double Dual Spaces

If V is a vector space, then so is the dual space V^*. Hence, we may form the double dual space $(V^*)^*$.

Definition 4.2.1. Let V be a vector space over a field F. The *double dual space* of V, denoted by V^{**}, is defined as

$$V^{**} = \mathcal{L}(V^*, F) = \{\bar{v} : V^* \to F \mid \bar{v} \text{ is a linear functional}\}.$$

What are elements of V^{**}? If $v \in V$, consider the map $\bar{v} : V^* \to F$ defined by

$$\bar{v}(\phi) = \phi(v) \quad \text{for all } \phi \in V^*.$$

This map is called the *evaluation at* v. To see that $\bar{v} \in V^{**}$, we must show that it is linear. But if $\alpha, \beta \in F$, $\phi, \psi \in V^*$, then

$$\bar{v}(\alpha\phi + \beta\psi) = (\alpha\phi + \beta\psi)(v) = \alpha\phi(v) + \beta\psi(v) = \alpha\bar{v}(\phi) + \beta\bar{v}(\psi)$$

for all $v \in V$, and so $\bar{v}$ is indeed linear.

Since the evaluation at v is in V^{**} for all $v \in V$, we can define a map $\tau : V \to V^{**}$ by

$$\tau(v) = \bar{v} \quad \text{for all } v \in V.$$

This is called the *canonical map* (or the *natural map*) from V to V^{**}.

Theorem 4.2.2. *Let V be a vector space over a field F.*

(i) For any non-zero vector $v \in V$, there exists a linear functional $\phi \in V^*$ for which $\phi(v) \neq 0$.

(ii) A vector $v \in V$ is zero if and only if $\phi(v) = 0$ for all $\phi \in V^*$.

Theorem 4.2.3. *Let V be a vector space over a field F. Then the canonical map τ is a monomorphism. Furthermore, if V is finite-dimensional, then τ is an isomorphism.*

Corollary 4.2.4. *Let V be a finite-dimensional vector space over a field F. If $f \in V^{**}$, then there exists a unique vector $v \in V$ such that*

$$f(\phi) = \phi(v) \qquad \text{for all } \phi \in V^*.$$

Let V be a **finite-dimensional** vector space. Then

$$V \cong V^* \cong V^{**}.$$

Thus, we can conclude that $V \cong V^{**}$ by $v \mapsto \bar{v}$. That is, we can **identify** the double dual space V^{**} with V. Alternatively, we say that V **is** the **dual space** of V^*, or the spaces V, V^* are naturally in duality with one another. Each is the dual space of the other.

Is every basis of V^* the dual basis of some basis of V? Equivalently,

$$\forall\ B^* \subseteq V^* \quad B^* \text{ is a basis of } V^* \implies \exists \text{ a basis } B \text{ of } V \text{ such that } B^* \text{ is the dual basis of } B$$

Proposition 4.2.5. *Let V be a finite-dimensional vector space over a field F. Then each basis of V^* is the dual basis of some basis of V.*

Let V be a **finite-dimensional** vector space. Then

V^* is the dual space of V with B^* is the dual basis of a basis B and
V "**is**" the dual space of V^* with B "**is**" the dual basis of a basis B^*.