2301610 Linear and Multilinear Algebra

EXERCISE 2.1

1. Let V and W be finite-dimensional vector spaces, and let T be a linear transformation from V to W. Prove or disprove the followings.

(1.1) If $T(v) = 0$ only when $v = 0$, then $\dim V = \dim W$.

(1.2) If $\dim V = \dim \operatorname{im} T$, then $\ker T = \{0\}$.

(1.3) $\ker T^2 \supseteq \ker T$.

(1.4) $\dim \ker T \leq \dim \operatorname{im} T$.

2. If T is a linear transformation on an n-dimensional vector space V, and if $T^{n-1}(x) \neq 0$, but $T^n(x) = 0$, show that $x, T(x), T^2(x), \ldots, T^{n-1}(x)$ are linearly independent. Find the matrix of T with respect to this basis.

3. Let V be an n-dimensional vector space over F. Show that $V \cong F^n$.

4. Let W be a subspace of a finite-dimensional vector space V. If T is a linear transformation from W to V, then prove or disprove that T can be extended to a linear transformation on V.

5. Let T be a linear transformation on $\mathbb{R}^3$ such that

$$T(1,0,1) = (2,3,-1), \quad T(1,-1,1) = (3,0,-2) \quad \text{and} \quad T(-2,7,-1) = (2,3,-1).$$

Find $\operatorname{im} T$ and write the formula of T.

6. Let $B = \{e_1, e_2, e_3, e_4\}$ be a basis for a vector space V of dimension 4, and let T be a linear transformation on V having its matrix respect to the basis

$$m_B[T] = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & 5 & 5 \\ 2 & -2 & 1 & -2 \end{bmatrix}$$

(6.1) Find $\ker T$ and $\operatorname{im} T$.

(6.2) Take a basis for $\ker T$, extend it to a basis for V, then find the matrix of T with respect to this basis.

7. Let V and W be vector spaces over the same field F. Define an addition on $\mathcal{L}(V,W)$ and a scalar multiplication as follows:

$$(f+g)(v) = f(v) + g(v) \qquad \text{for all } f, g \in \mathcal{L}(V,W),\ v \in V,$$
$$(\alpha f)(v) = \alpha f(v) \qquad \text{for all } f \in \mathcal{L}(V,W),\ \alpha \in F,\ v \in V.$$

Prove that $\mathcal{L}(V,W)$ is a vector space over F.

8. Let V be an n-dimensional and W an m-dimensional vector spaces over the same field F. Prove that $\mathcal{L}(V,W) \cong M_{mn}(F)$ and $\dim \mathcal{L}(V,W) = mn$.

DUE THURSDAY 28 June 2007 before 15:00

Choose 4 problems from 8