2301610 Linear and Multilinear Algebra

EXERCISE 3

1. Prove the **Second Isomorphism Theorem**:

Let V be a vector space, and let U and W be subspaces of V. Then

$$\frac{U+W}{U} \cong \frac{W}{U \cap W}.$$

2. Let V be a finite-dimensional vector space and $W_1, \dots, W_n$ be subspaces of V such that $V = W_1 \oplus \cdots \oplus W_n$. Prove that $\dim V = \dim W_1 + \cdots + \dim W_n$.

3. Prove that a finite set $\{v_1, \dots, v_n\}$ of non-zero vectors in a vector space V is a basis of V if and only if

$$V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle.$$

4. Let U be a subspace of a vector space V. Prove that any complement of U in V is isomorphic to the quotient space V/U.

5. Let V a finite-dimensional vector space and suppose that $V = U \oplus W_1$ and $V = U \oplus W_2$. What can you say about the relationship between W_1 and W_2?

6. If $\tau \in \mathcal{L}(V_1, W_1)$ and $\sigma \in \mathcal{L}(V_2, W_2)$, we define the *external direct sum*

$$\tau \boxplus \sigma : V_1 \oplus V_2 \to W_1 \oplus W_2 \text{ by } \big(\tau \boxplus \sigma\big)\big(v_1 + v_2\big) = \tau(v_1) + \sigma(v_2),$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Prove that $\tau \boxplus \sigma$ is a linear transformation.

7. Find a vector space V, a subspace W of V, and projections P_1 and P_2 of V onto W such that $P_1 \neq P_2$.

8. Let V be a vector space and suppose that $P_1, \dots, P_n : V \to V$ are linear transformations such that

(i) $P_1 + \cdots + P_n = 1_V$, and

(ii) $P_i \circ P_j = 0$ for all $i \neq j$.

Prove that P_i is a projection for all $i \in \{1, \dots, n\}$.

9. Let T be a linear transformation from V onto W. Show that the map

$$U \mapsto T^{-1}(U) = \{v \in V \mid T(v) \in U\},$$

where $U \preceq W$, is a one-to-one correspondence between the subspaces of W and the subspaces of V which contain $\ker T$.

DUE TUESDAY 17 July 2007 before 15:00

Choose 4 problems from 9