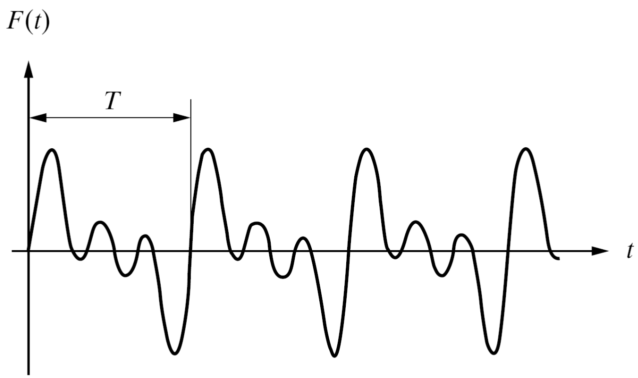
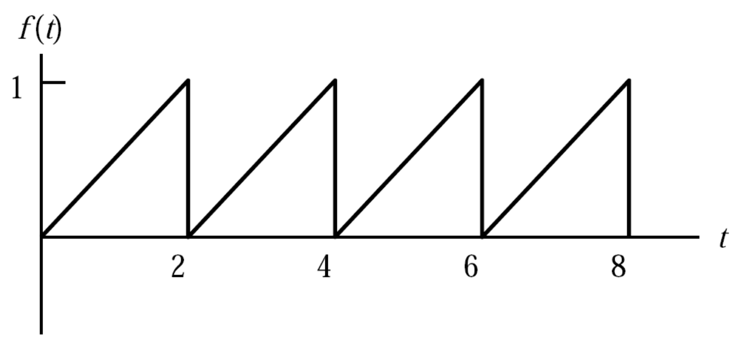
Periodic excitations

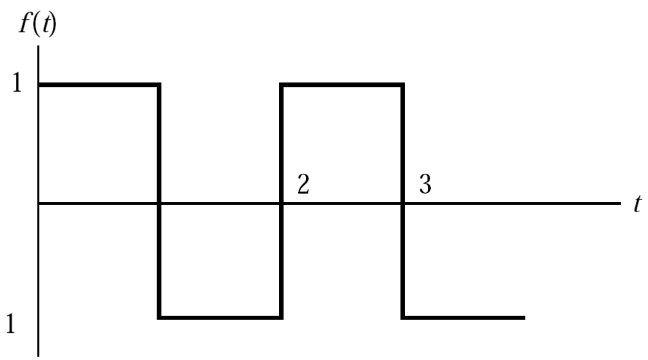
Periodic excitation repeats every T second (period)

Periodic function: $f(t) = f(t+T)$



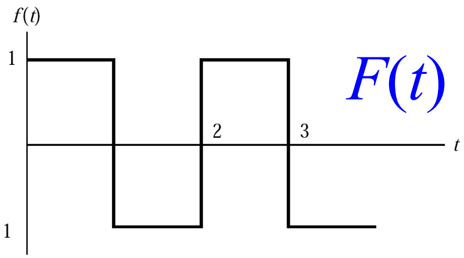
<u>Example</u>: vibration of rotating machines ex. engine, gear, …





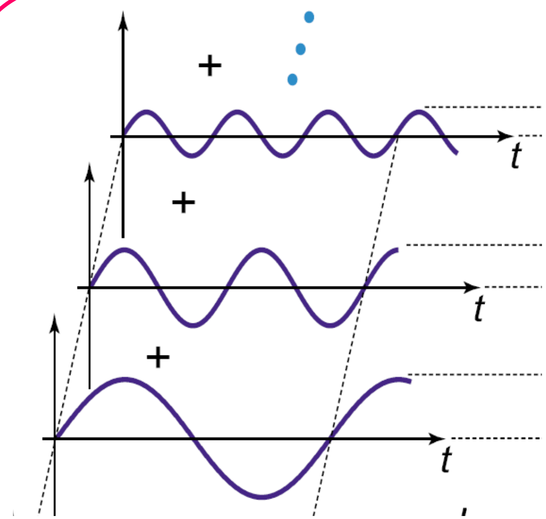
Basic concept

Periodic excitation



$F(t)$

Fourier series



Sum of harmonic excitations

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

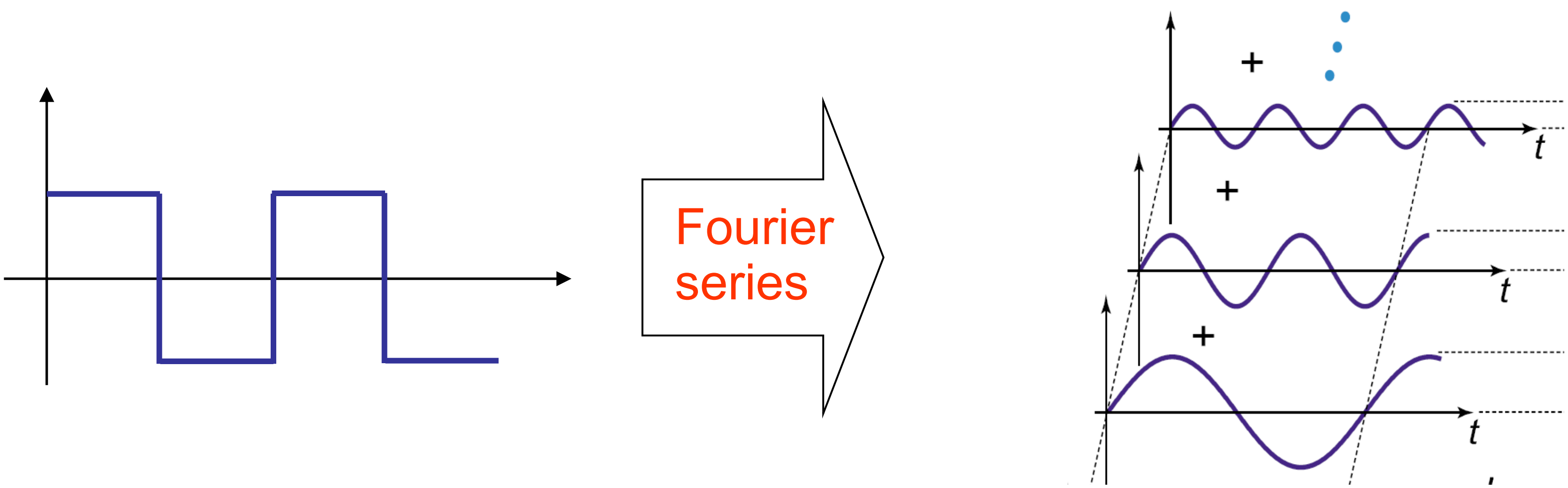
Superposition

Sum of harmonic response $x(t) = x_1 + x_2 + x_3 + \dots$

Fourier series (1)

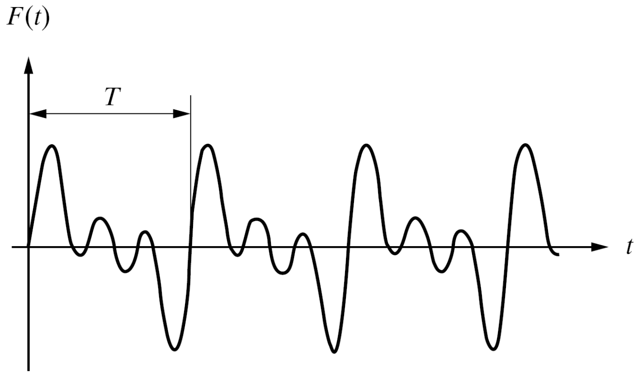
Concept: Any periodic function can be expressed as linear combinations of harmonic functions whose frequencies are multiples of the fundamental frequency

$$F(t) = C + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$
$$+ b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \ldots$$



Fourier series (2)

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t)$$



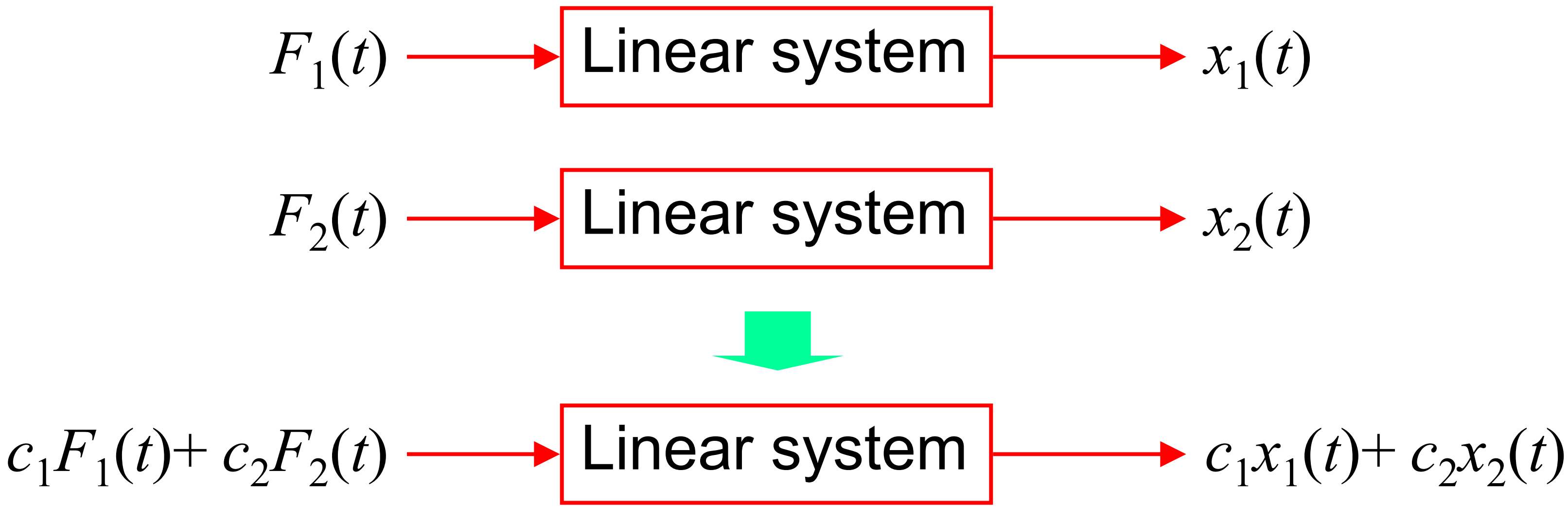
Where $$\omega_T = \frac{2\pi}{T}$$

$$a_0 = \frac{2}{T} \int_0^T F(t) dt$$

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\omega_T t \, dt$$

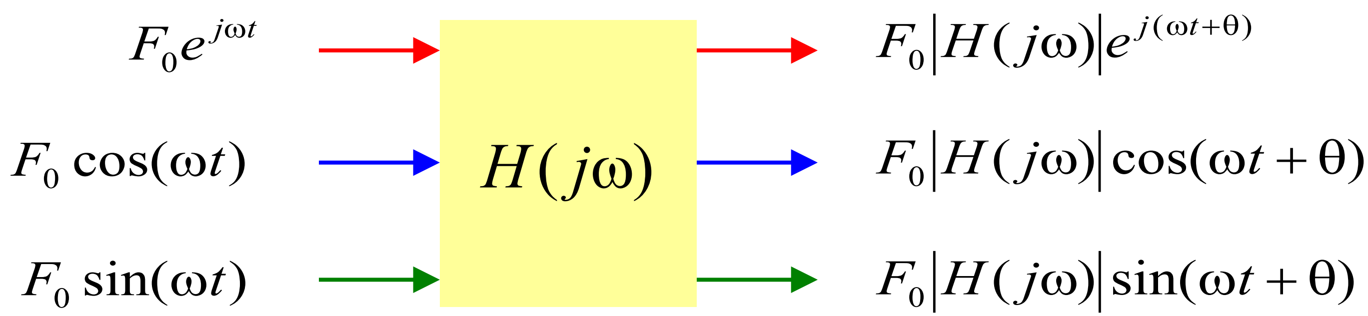
$$b_n = \frac{2}{T} \int_0^T F(t) \sin n\omega_T t \, dt$$

Superposition principle

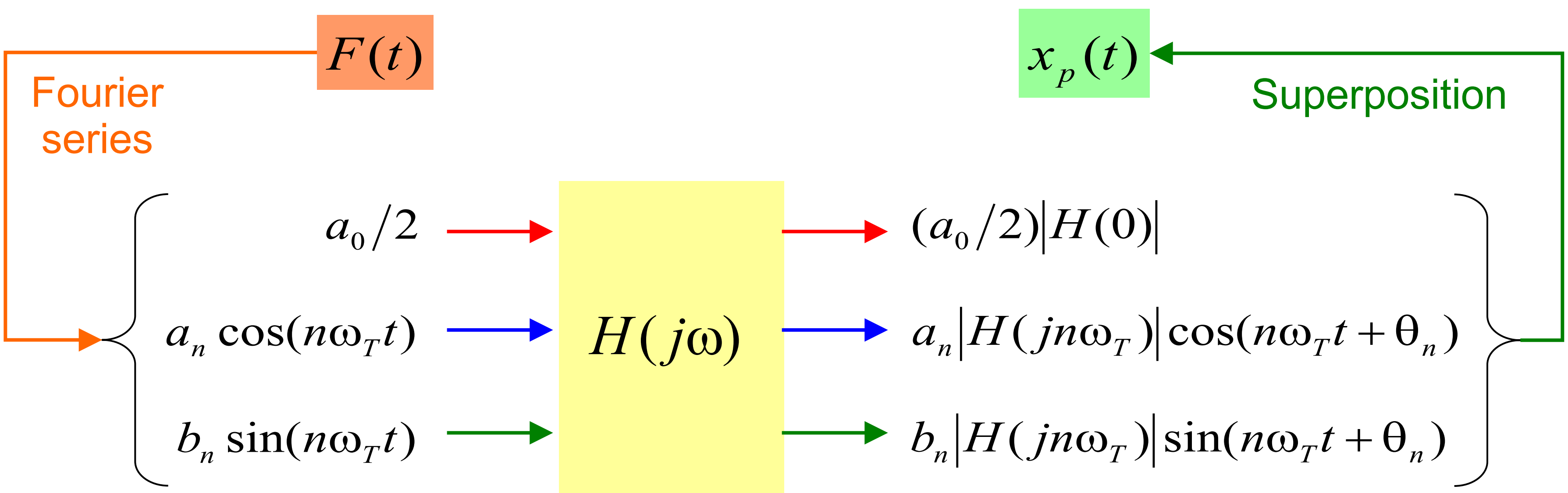


Response to periodic excitation

Harmonic excitation (review)



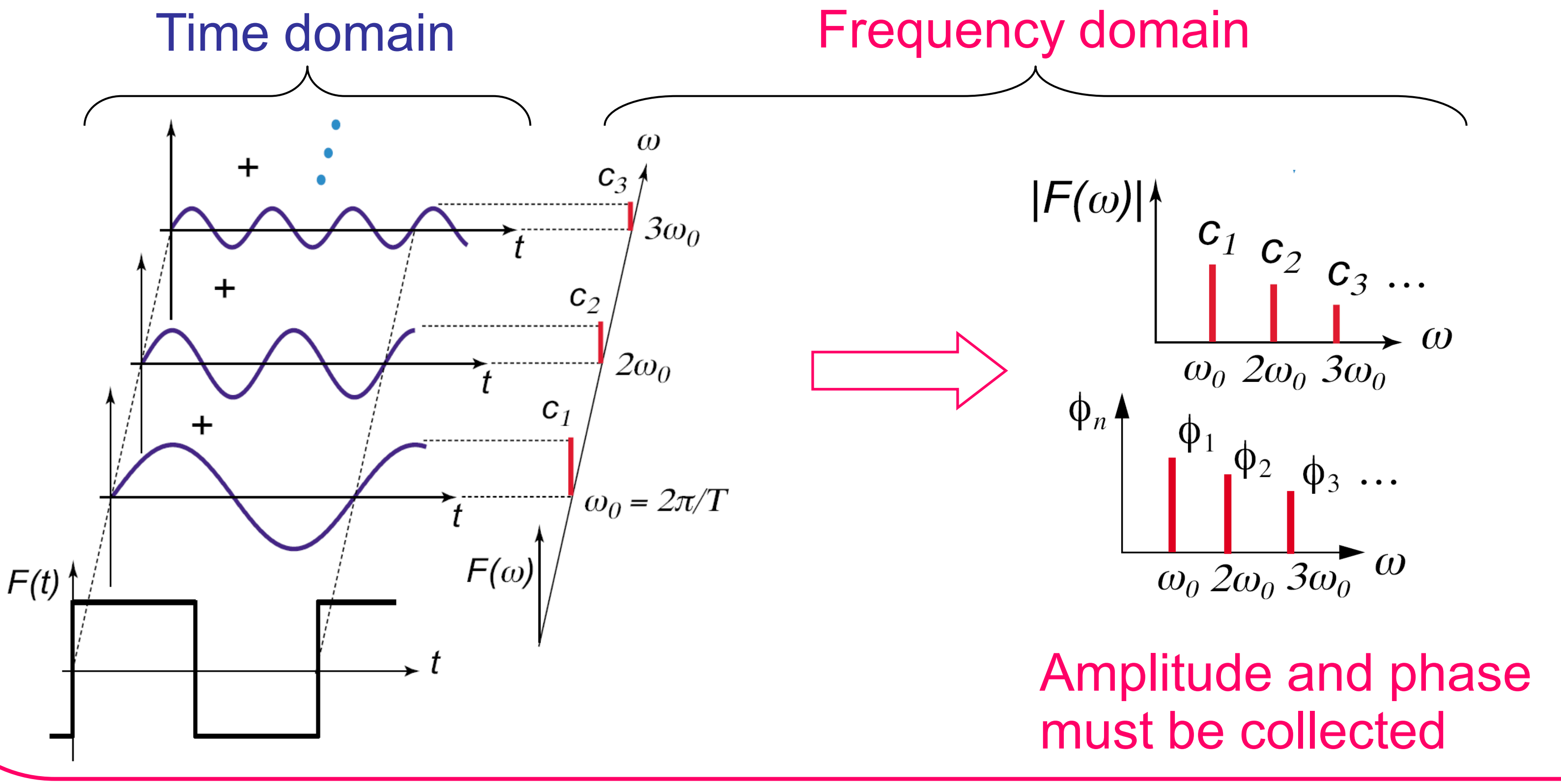
Periodic excitation



Frequency domain

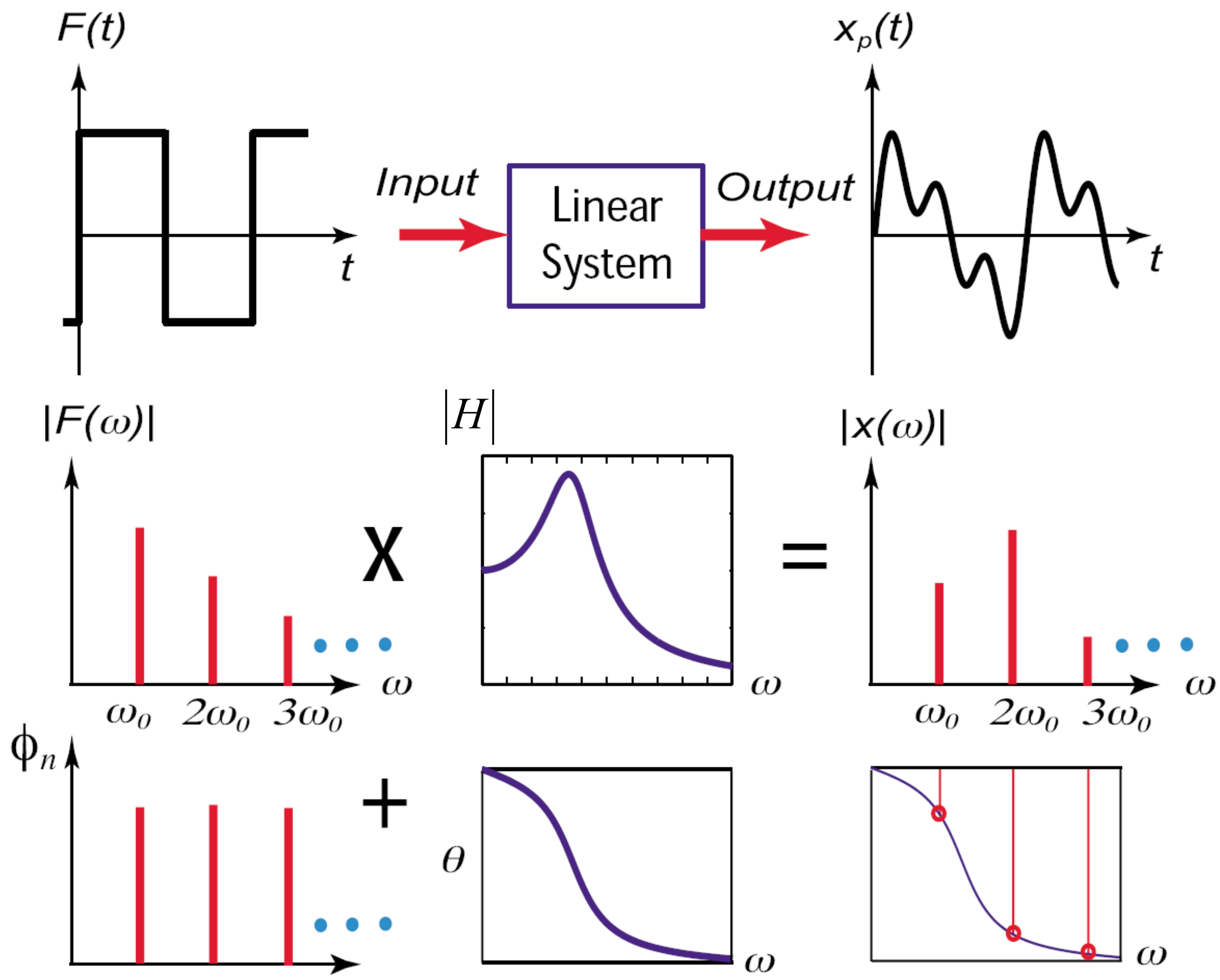
Time domain vs Frequency domain

- Interchangeable (no information lost)
- The frequency domain representation of a signal is called the "spectrum" of the signal



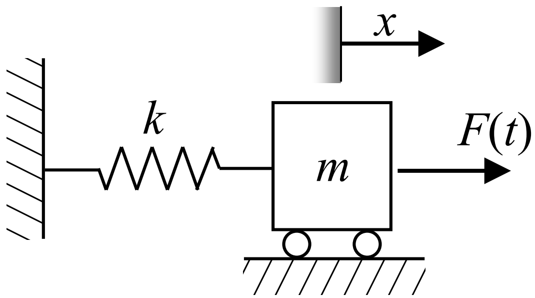
Amplitude and phase must be collected

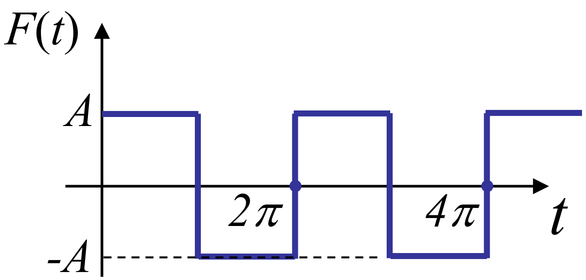
Response to excitation (freq. domain)



Example (Periodic excitation)-1

A square wave force $F(t)$ is applied to a 1-kg mass with k = 4 N/m. Determine steady-state response of the mass.





Solution

EOM: $m\ddot{x} + kx = F(t) \Longrightarrow \ddot{x} + 4x = F(t) \qquad (\omega_n = 2)$

The exciting force can be written as $F(t) = \begin{cases} A; & 0 \le t < T/2 \\ -A; & T/2 \le t < T \end{cases}$

where $T = 2\pi \Longrightarrow \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ rad/s

Example (Periodic excitation)-2

The exciting force can be written as $F(t) = \begin{cases} A; & 0 \le t < T/2 \\ -A; & T/2 \le t < T \end{cases}$

Fourier series $$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_T t + b_n \sin n\omega_T t)$$

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\omega_T t \, dt = 0$$

$$b_n = \frac{2}{T} \int_0^T F(t) \sin n\omega_T t \, dt = \frac{2A}{n\pi}(1 - \cos n\pi)$$

Therefore $$F(t) = \frac{4A}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \ldots \right)$$

Example (Periodic excitation)-3

EOM: $\ddot{x} + 4x = F(t) \quad \Longrightarrow \quad \ddot{x} + 4x = \dfrac{4A}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \dots\right)$

The steady-state response is

$$x_p = \frac{4A}{\pi}\left(\left|H(1)\right|\sin(t+\theta_1) + \frac{1}{3}\left|H(3)\right|\sin(3t+\theta_3) + \frac{1}{5}\left|H(5)\right|\sin(5t+\theta_5) + \dots\right)$$

Review

EOM: $m\ddot{x} + kx = F_0 \sin \omega t$

The steady-state response is $x_p(t) = \left|H(\omega)\right| F_0 \cdot \sin(\omega t + \theta)$

where $\left|H(\omega)\right| = \dfrac{1}{\left|\omega_n^2 - \omega_T^2\right|}$

$\theta = 0$ when $\omega_T < \omega_n$

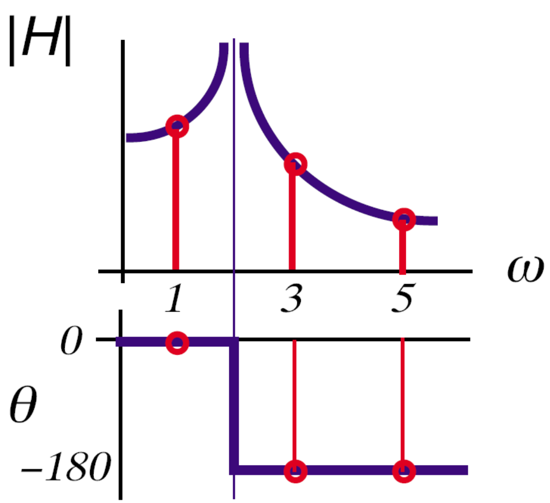
$\theta = -\pi$ when $\omega_T > \omega_n$

Example (Periodic excitation)-4

EOM: $\ddot{x} + 4x = \frac{4A}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$

The steady-state response is

$$x_p = \frac{4A}{\pi}\left(|H(1)|\sin(t+\theta_1) + \frac{1}{3}|H(3)|\sin(3t+\theta_3) + \frac{1}{5}|H(5)|\sin(5t+\theta_5) + \ldots\right)$$



n	$\lvert H(nj)\rvert$	θ_n
1	1/3	0
3	1/5	-180
5	1/21	-180