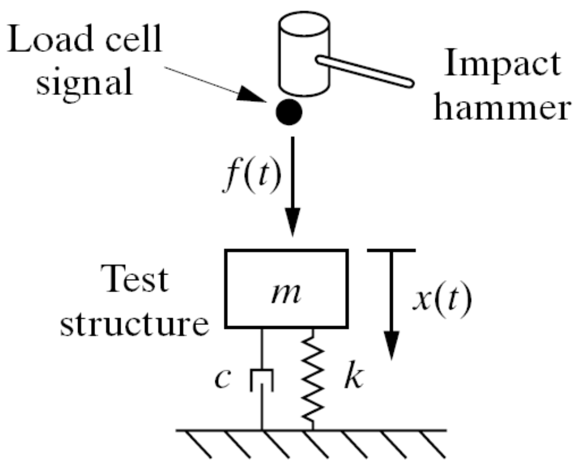
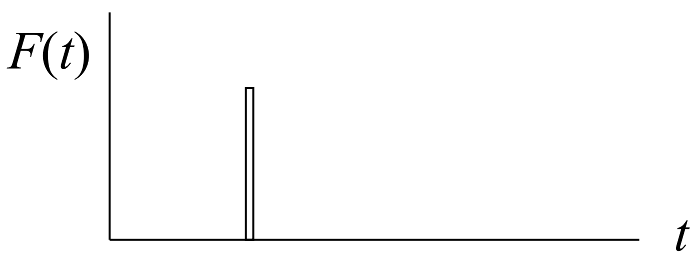
General forced response

Impulse response

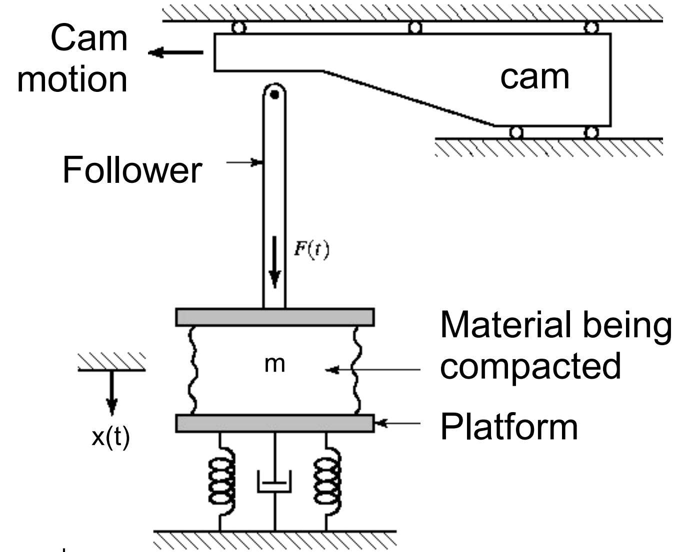
- Dirac delta function
- Unit impulse response

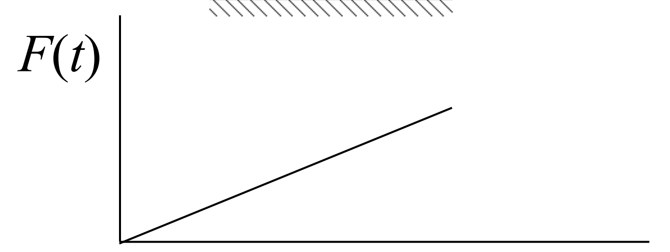




arbitrary response

- Convolution integral





The principle of Superposition

Linear equations

$\ddot{x} + \omega_n^2 x = 0$ ⇒ x_1, x_2 are both solutions

⇒ $x = a_1 x_1 + a_2 x_2$ is also a solution

a_1, a_2 are constants

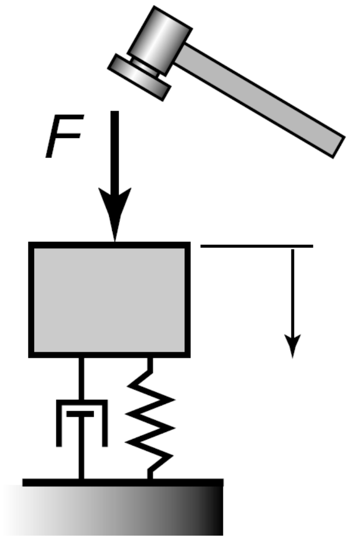
$\ddot{x} + \omega_n^2 x = f_1$ ⇒ solution x_1

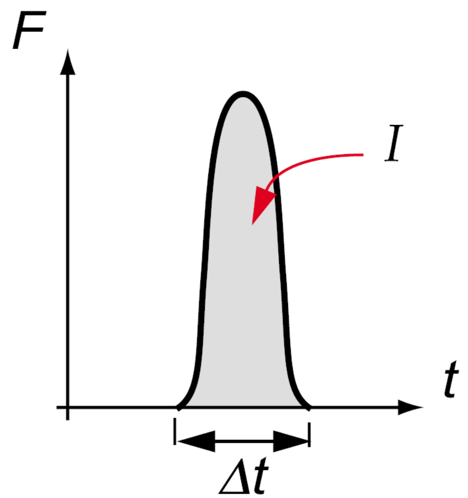
$\ddot{x} + \omega_n^2 x = f_2$ ⇒ solution x_2

⇒ $\ddot{x} + \omega_n^2 x = f_1 + f_2$ ⇒ solution $x_1 + x_2$

Impact excitation

Impact is a force applied for a very short period of time



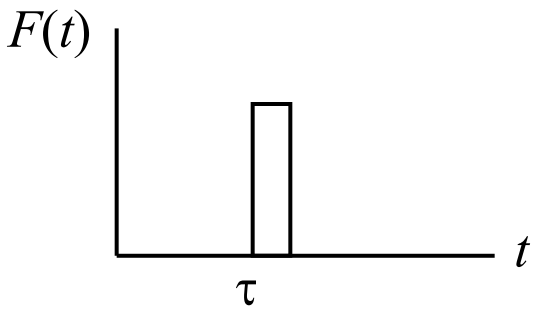


Impulse due to $F(t)$: $I = \int F(t)\,dt$

$=$ area under F-t curve

Dirac delta function

Dirac delta function is a unit impulse function

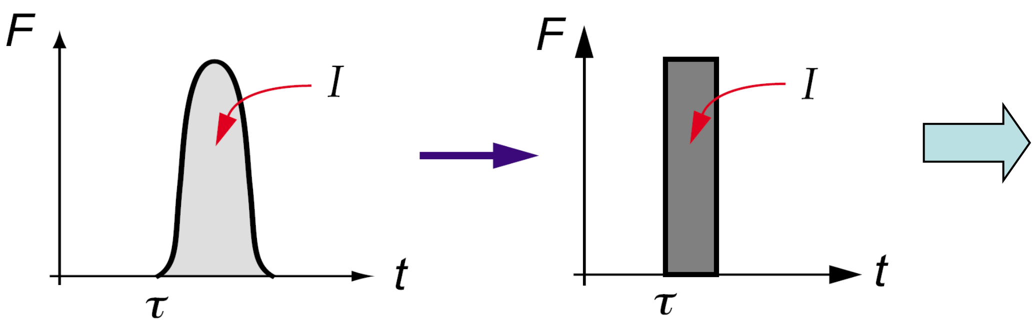


Properties of dirac delta function

$$\delta(t-\tau)=0 \quad \text{for} \quad t\neq\tau$$

$$\int_{-\infty}^{\infty}\delta(t-\tau)dt=1$$

$$\int_{-\infty}^{\infty}g(t)\cdot\delta(t-\tau)dt=g(\tau)$$



$$F(t)=I\cdot\delta(t-\tau)$$

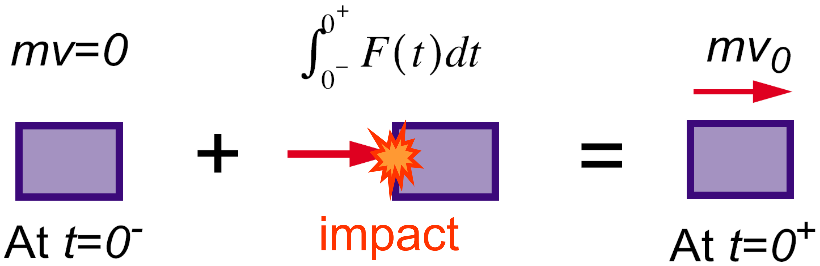
Impulse

Linear impulse-momentum equation

$$F = ma \quad \Rightarrow \quad \int F\,dt = \int (ma)\,dt$$

$$I = m(v_2 - v_1) = m\Delta v$$

A mass m initially at rest subject to an impact of size I N.s

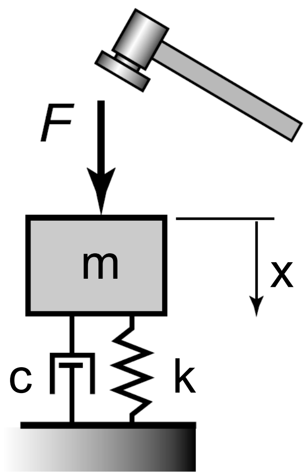


$$\int F\,dt = I = mv_0$$

$$v_0 = I/m$$

An impact causes initial velocity change of $v_0 = I/m$ while keeping initial displacement unchanged

Unit impulse response (1)



Time response of a system initially at rest but subject to a unit impact ($I = 1$)

EOM $$m\ddot{x} + c\dot{x} + kx = \delta(t)$$

Initial conditions $$x(0) = \dot{x}(0) = 0$$

Just after subject to a unit impact ($t = 0^+$)

EOM $$m\ddot{x} + c\dot{x} + kx = 0$$

Initial conditions $$x(0) = 0; \quad \dot{x}(0) = 1/m$$

Impact response is just free vibration of a system with initial velocity 1/m

Unit impulse response (2)

EOM $m\ddot{x} + c\dot{x} + kx = 0$

Initial conditions $x(0) = 0; \quad \dot{x}(0) = 1/m$

Unit impulse response is just free vibration of a system with initial velocity $1/m$

underdamped $$h(t) = x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

For impact applied at time $t = \tau$

$$h(t) = \begin{cases} 0 & \text{for } 0 < t < \tau \\ \dfrac{1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau) & \text{for } t > \tau \end{cases}$$

Before impact

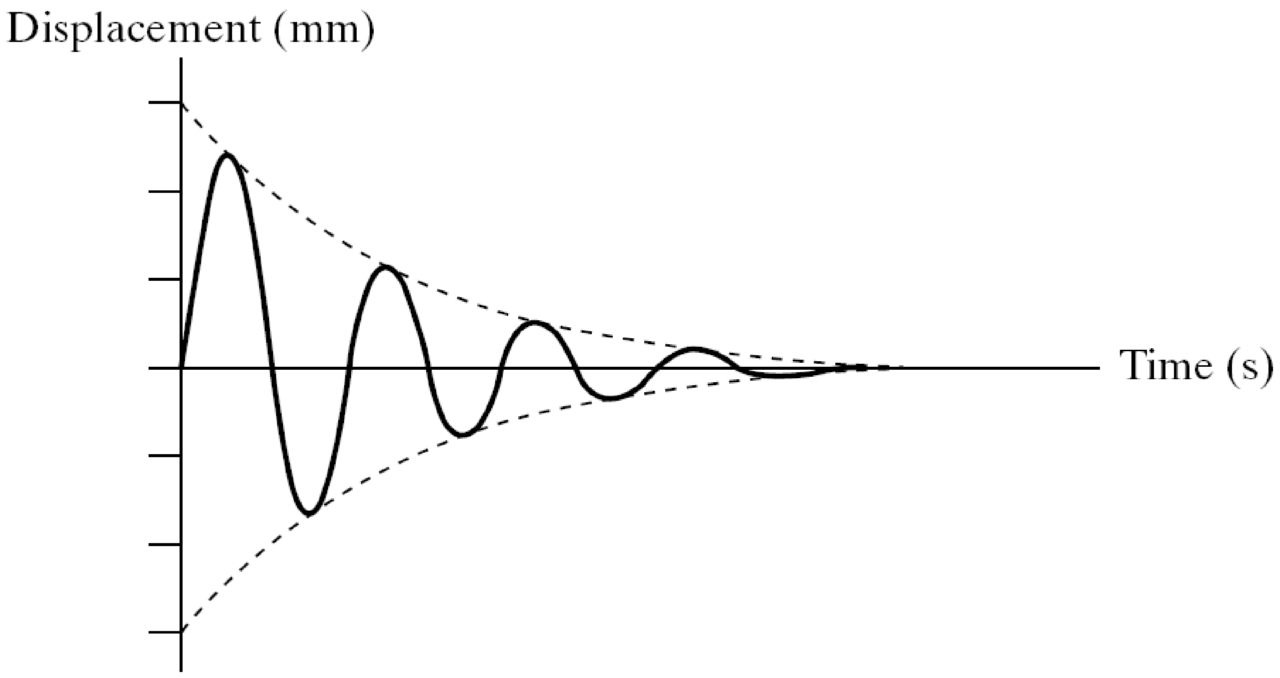
After impact

Unit impulse response (3)

EOM $$m\ddot{x} + c\dot{x} + kx = 0$$

Initial conditions $$x(0) = 0; \quad \dot{x}(0) = 1/m$$

underdamped $$h(t) = x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$



Impulse response

EOM $m\ddot{x} + c\dot{x} + kx = I\delta$

Initial cond. $x(0) = 0; \quad \dot{x}(0) = 0$

$$\left.\begin{array}{l} m\ddot{x} + c\dot{x} + kx = I\delta \\ x(0) = 0; \quad \dot{x}(0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} m\ddot{x} + c\dot{x} + kx = 0 \\ x(0) = 0; \quad \dot{x}(0) = I / m \end{array}$$

Unit impulse response is just free vibration of a system with initial velocity I / m

underdamped

$$x(t) = \frac{I}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t = I \cdot h(t)$$

For impact applied at time $t = \tau$

$$h(t) = \begin{cases} 0 & \text{for } 0 < t < \tau \\ \dfrac{I}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t - \tau) & \text{for } t > \tau \end{cases}$$

Before impact

After impact

Example 1-1

Given: $m = 1$ kg, $c = 0.5$ kg/s, $k = 4$ N/m

$$F(t) = 0.2\delta(t) + 0.1\delta(t-\tau)$$

$$\omega_n = \sqrt{4} = 2 \quad ; \quad \zeta = c/(2m\omega_n) = 0.125$$

For $F(t) = 0.2\delta(t)$

$$x_1(t) = \frac{0.2}{(1)(2\sqrt{1-0.125^2})} e^{-(0.125)(2)t} \sin 2\sqrt{1-0.125^2} \cdot t$$

$$x_1(t) = 0.1008e^{-0.25t} \sin(1.984t)$$

- -

For $F(t) = 0.1\delta(t-\tau)$ $\quad t > \tau$

$$x_2(t) = 0.0504e^{-0.25(t-\tau)} \sin 1.984(t-\tau)$$

Example 1-2

$$x(t) = x_1(t) + x_2(t)$$

$$= \begin{cases} 0.1008e^{-0.25t}\sin(1.984t) & 0 < t < \tau \\ 0.1008e^{-0.25t}\sin(1.984t) + 0.0504e^{-0.25(t-\tau)}\sin 1.984(t-\tau) & t > \tau \end{cases}$$

⇩

$$x(t) = 0.1008e^{-0.25t}\sin(1.984t) + \left[0.0504e^{-0.25(t-\tau)}\sin 1.984(t-\tau)\right]\cdot\Phi(t-\tau)$$

Heaviside step function

$$\Phi(t-\tau) = \begin{cases} 0, & t < \tau \\ 1, & t \geq \tau \end{cases}$$

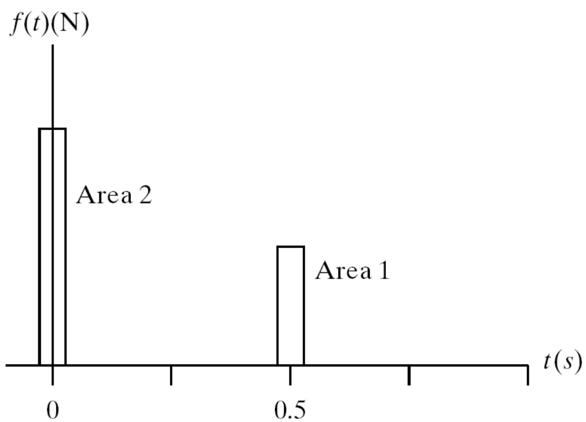
or $H(t-\tau)$

Example 1-3

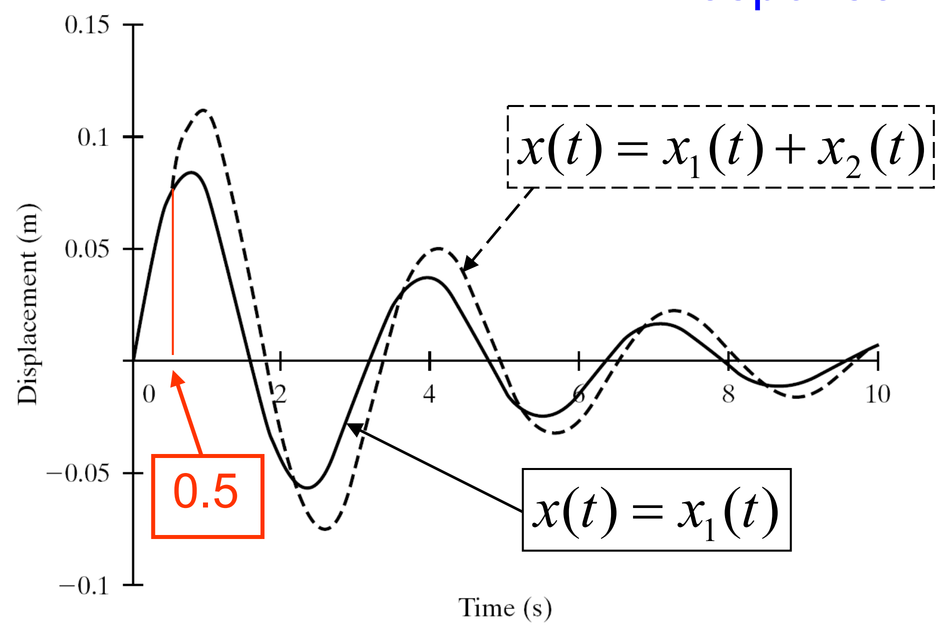
$$x(t) = x_1(t) + x_2(t)$$

$$= \begin{cases} 0.1008e^{-0.25t}\sin(1.984t) & 0 < t < \tau \\ 0.1008e^{-0.25t}\sin(1.984t) + 0.0504e^{-0.25(t-\tau)}\sin 1.984(t-\tau) & t > \tau \end{cases}$$

Impact forces $\tau = 0.5$



Response

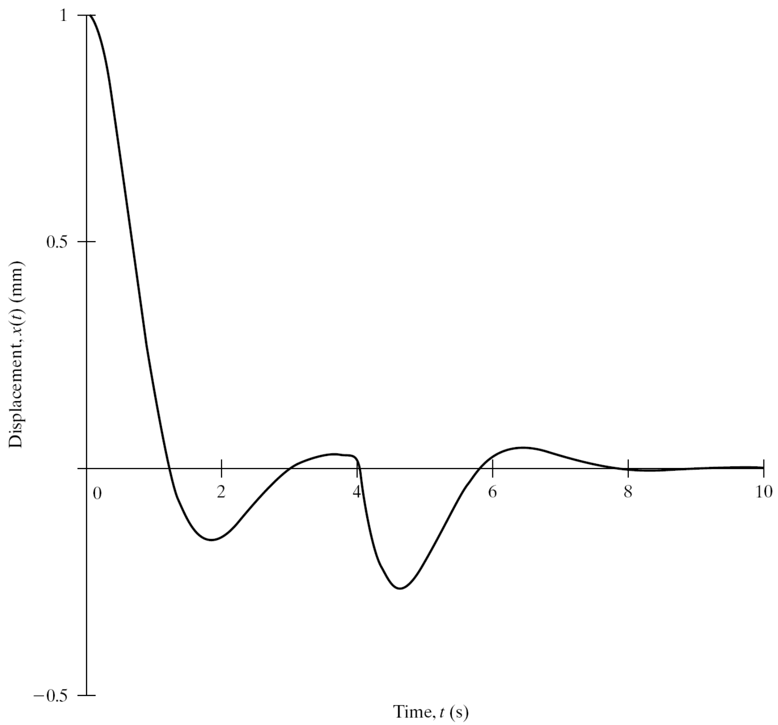


Example 2-1

$$\ddot{x}(t) + 2\dot{x}(t) + 4x(t) = \delta(t) - \delta(t-4)$$

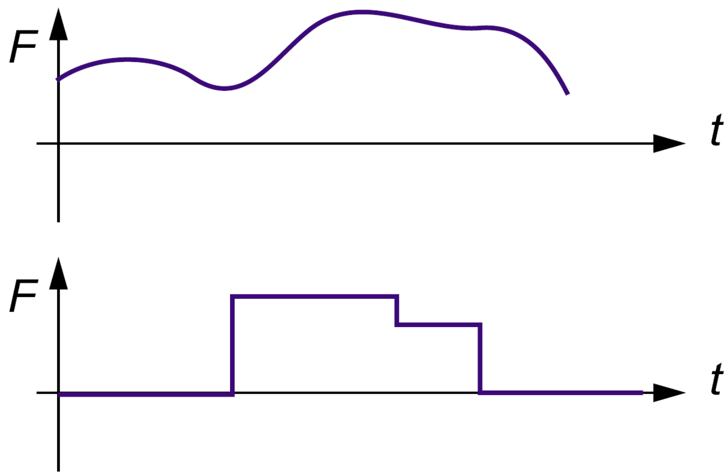
$$x_0 = 1\,\text{mm}; \quad v_0 = -1\,\text{mm/s}$$

Compute and plot the response



Arbitrary input

Vibration of systems subject to arbitrary nonperiodic inputs.



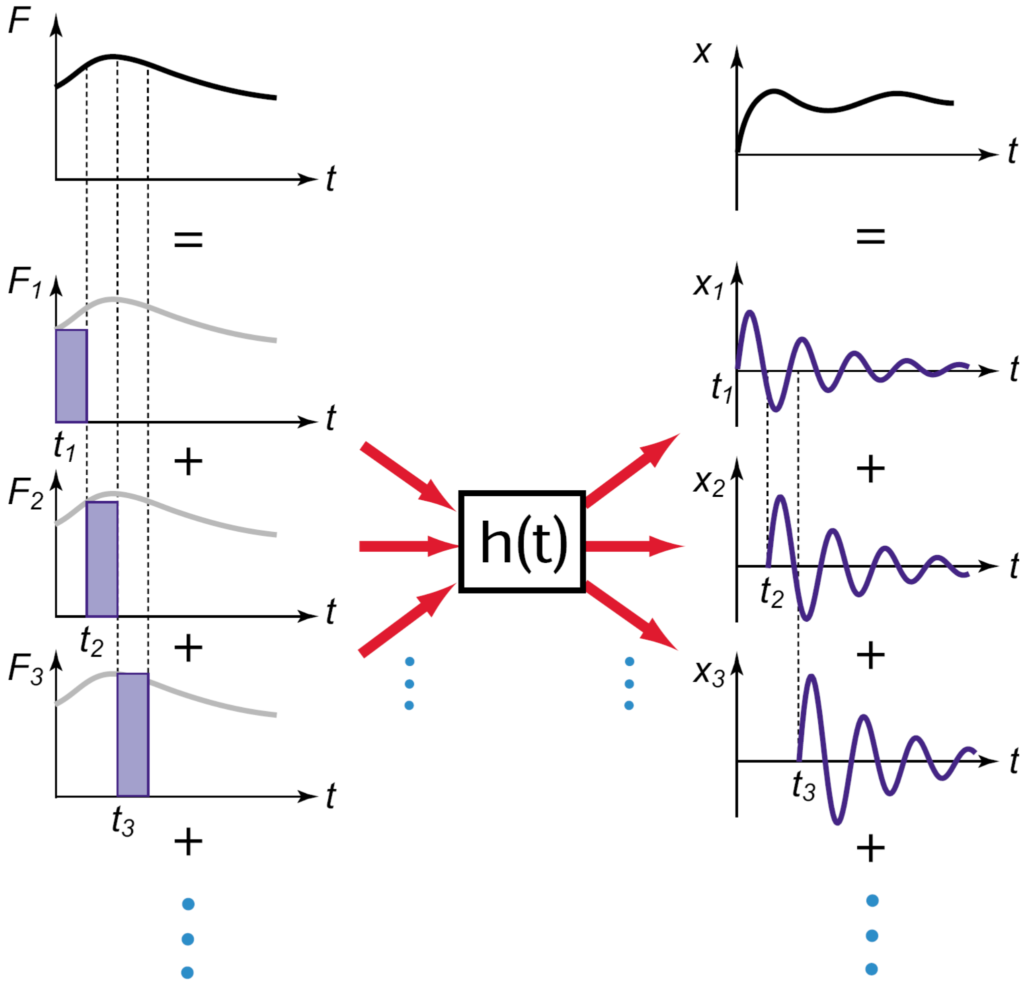
Analysis tools:

- Impulse response function
- Super position
- Convolution integral

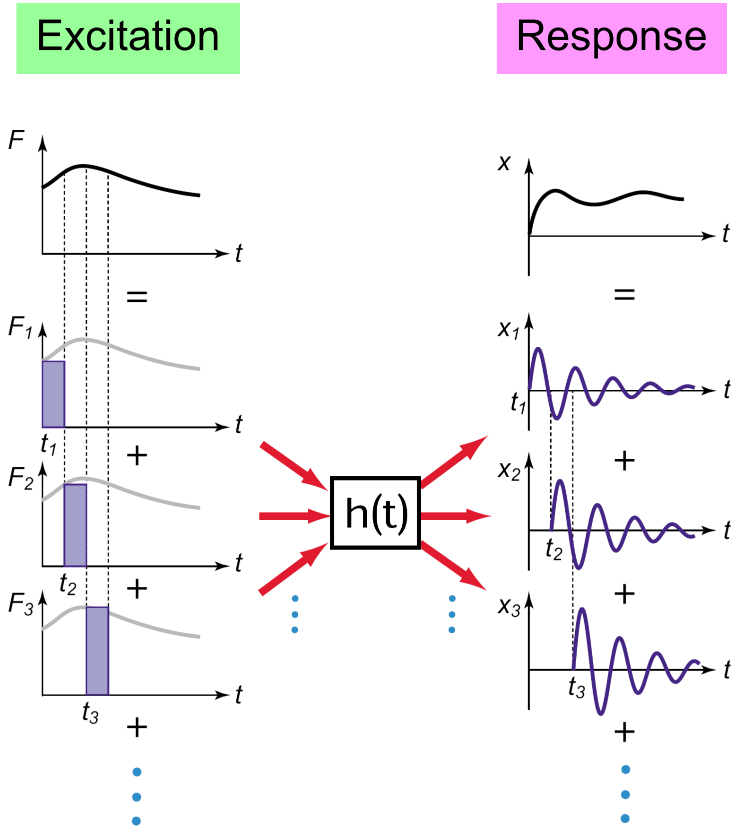
Convolution integral (1)

Excitation

Response



Convolution integral (2)



$$x(t) = x_1 + x_2 + x_3 + \ldots$$

$$x(t) = I_1 h(t - t_1) + I_2 h(t - t_2) + I_3 h(t - t_3) + \ldots$$

$$x(t) = F_1 \Delta t \cdot h(t - t_1) + F_2 \Delta t \cdot h(t - t_2) + F_3 \Delta t \cdot h(t - t_3) + \ldots$$

$$\Delta t \to 0$$

$$x(t) = \int_0^t F(\tau) h(t - \tau) d\tau$$

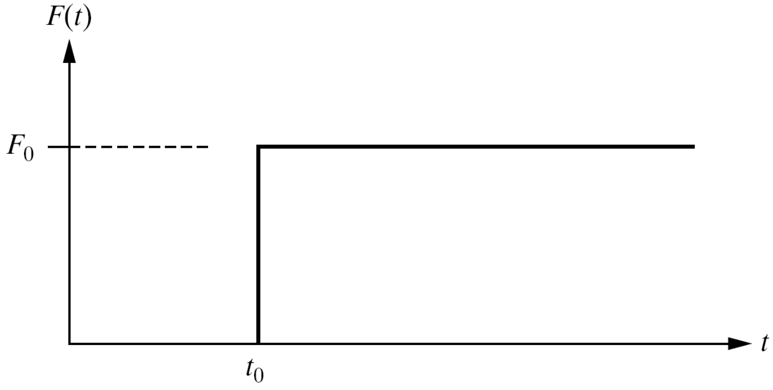
Convolution integral (3)

For the non-continuous function,

$$m\ddot{x} + c\dot{x} + kx = \begin{cases} F_1(t), & 0 \le t < t_1 \\ F_2(t), & t \ge t_1 \end{cases}$$

$$x(t) = \begin{cases} \int_0^{t_1} F_1(\tau)h(t-\tau)d\tau & \text{when} \quad 0 \le t < t_1 \\ \int_0^{t_1} F_1(\tau)h(t-\tau)d\tau + \int_{t_1}^{t} F_2(\tau)h(t-\tau)d\tau & \text{when} \quad t \ge t_1 \end{cases}$$

Example 3-1



$$m\ddot{x} + c\dot{x} + kx = \begin{cases} 0, & 0 < t < t_0 \\ F_0, & t \geq t_0 \end{cases}$$

$$x_0 = v_0 = 0$$

$0 < \zeta < 1$ (underdamped)

$$x(t) = \int_0^t F(\tau) h(t-\tau) d\tau$$

$$h(t) = x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$x(t) = \int_{t_0}^{t} F_0 \frac{1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

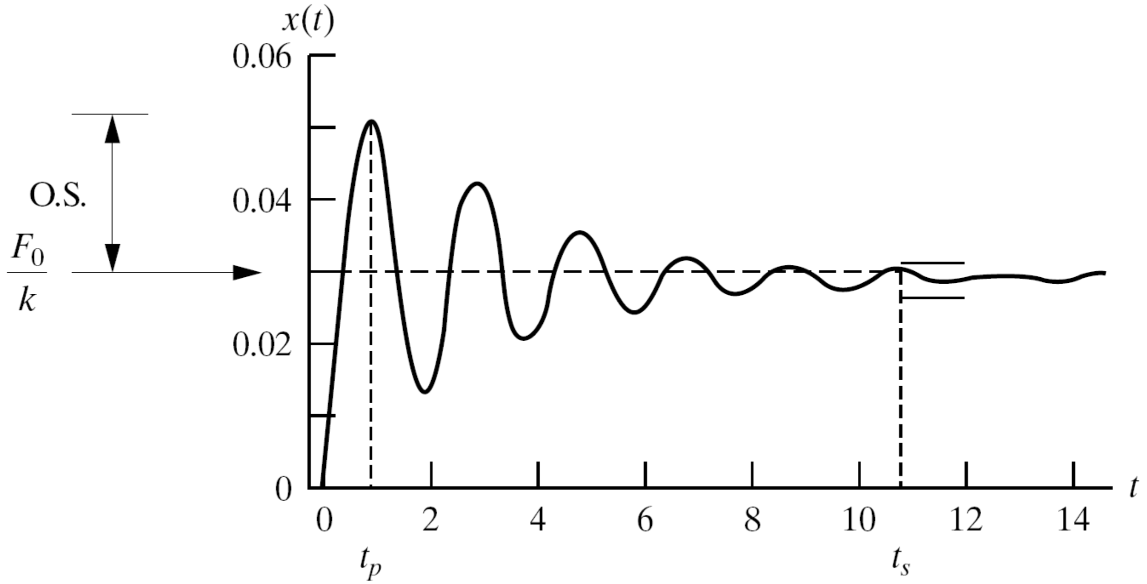
$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_{t_0}^{t} e^{\zeta\omega_n \tau} \sin \omega_d (t-\tau)\, d\tau$$

Example 3-2

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n (t-t_0)} \cos[\omega_d (t-t_0) - \theta]; \qquad t \ge t_0$$

Static displacement

$$\theta = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}$$



Example

$\zeta = 0.1$

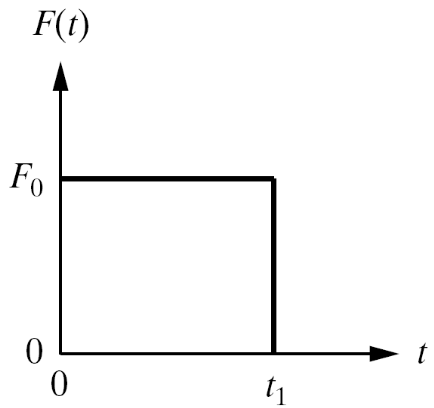
$\omega_n = 3.16$ rad/s

$F_0 = 30$ N

$k = 1000$ N/m

$t_0 = 0$

Example 4-1



$$m\ddot{x} + c\dot{x} + kx = \begin{cases} F_0, & 0 < t \le t_1 \\ 0, & t > t_1 \end{cases}$$

$$x_0 = v_0 = 0$$

$0 < \zeta < 1$ (underdamped)

Homework

Calculate and plot the response of an undamped system to a step function with a finite rise time of t_1 for the case m = 1 kg, k = 1 N/m, t_1 = 4 s, and F_0 = 20 N. This function is described by

$$F(t) = \begin{cases} \dfrac{F_0 t}{t_1}, & 0 \le t \le t_1 \\ F_0, & t > t_1 \end{cases}$$