Parametric Bootstrap in Small Area Estimation

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A Simple Two-level Model

Efron and Morris (JASA, 1975)

For $i = 1, \ldots, m,$

Level 1: (Sampling Distribution) $Y_i | \theta_i \overset{\text{iid}}{\sim} N(\theta_i, 1),$

Level 2: (Prior Distribution) $\theta_i \overset{\text{iid}}{\sim} N(\mu, A).$

The above model can be also viewed as a simple linear mixed model:

$$Y_i = \mu + v_i + e_i,$$

where $\{v_i\}$ and $\{e_i\}$ are independent with $v_i \overset{\text{iid}}{\sim} N(0, A)$ and $e_i \overset{\text{iid}}{\sim} N(0, 1)$ for $i = 1, \ldots, m.$
The best prediction (BP) (Bayes) estimator of $\theta_i$ under the above model and squared error loss:

$$\hat{\theta}_i^B = (1 - B) Y_i + B \mu,$$

where $B = \frac{1}{1 + A}$.

An empirical Bayes estimator of $\theta_i$ is given by

$$\hat{\theta}_i^{EB} = (1 - \tilde{B}) + \tilde{B} \bar{Y},$$

where $\bar{Y} = \frac{1}{m} \sum_{j=1}^{m} Y_j$,

$$\tilde{B} = \begin{cases} \hat{B} = \frac{m-3}{\sum_{j=1}^{m} (Y_j - \bar{Y})^2} & \text{if } \hat{B} < 1, \\ \frac{m-3}{m-1} & \text{otherwise.} \end{cases}$$

See Morris (1983).
Using a flat improper prior distributions on $\mu$ and $B$, Morris (1983) suggested an approximation to the posterior variance of $\theta_i$ as a measure of uncertainty of $\hat{\theta}_i^{EB}$. The measure is given by

$$V(\theta_i|y) = E \left[ V(\theta_i|y, \mu, B)|y \right] + V \left[ E(\theta_i|y, \mu, B)|y \right]$$

$$\approx (1 - \tilde{B}) + \frac{\tilde{B}}{m} + \frac{2\tilde{B}^2}{m - 3} (Y_i - \bar{Y})^2$$

$$= V_i^M \text{ (say)}$$
\[ V_{i}^{LL} = E_{*} V(\theta_{i} | y; \hat{\mu}^{*}, \hat{B}^{*}) + V_{*} E(\theta_{i} | y, \hat{\mu}^{*}, \hat{B}^{*}) \]
\[ \approx \frac{1}{R} \sum_{r=1}^{R} V(\theta_{i} | y; \hat{\mu}^{(r)}, \hat{B}^{(r)}) + \frac{1}{R - 1} \sum_{r=1}^{R} \left[ \hat{\theta}_{i}^{EB(r)} - \bar{\hat{\theta}}_{i}^{EB} \right]^{2} \]
\[ \approx (1 - \hat{B}_{0}) + \frac{m - 1}{m - 5} \frac{\tilde{B}_{0}}{m} + \frac{2\tilde{B}_{0}^{2}}{m - 5} (Y_{i} - \bar{Y})^{2}, \]

where

- \( V(\theta_{i} | y; \hat{\mu}^{(r)}, \hat{B}^{(r)}) = 1 - \hat{B}(r) \)
- \( \hat{\theta}_{i}^{EB(r)} = (1 - \hat{B}(r)) Y_{i} + \hat{B}(r) \hat{\mu}(r) \)
- \( \bar{\hat{\theta}}_{i}^{EB} = R^{-1} \sum_{r=1}^{R} \hat{\theta}_{i}^{EB(r)} \)

the estimates \( \hat{\mu}^{(r)} \) and \( \hat{B}^{(r)} \) of \( \mu \) and \( B \), respectively, are based on the \( r \)th parametric bootstrap sample (\( r = 1, \ldots, R \)).

- \( \hat{B}_{0} = \frac{m^{-1}}{m^{-3}} \tilde{B} \).
- The difference between \( V_{i}^{M} \) and \( V_{i}^{LL} \) is of order \( O_{p}(m^{-1}) \).
Taylor series linearization method due to Prasad and Rao (1990)

Define $MSE(\hat{\theta}_i^{EB}) = E(\hat{\theta}_i^{EB} - \theta_i)^2$, where the expectation is with respect to the joint distribution of $\{(Y_i, \theta_i), i = 1, \cdots, m\}$ under the linear mixed model. For this simple model, their second-order approximation to $MSE(\hat{\theta}_i^{EB})$ is given by

$$MSE(\hat{\theta}_i^{EB}) = g_1(B) + g_2(B) + g_3(B) + o(m^{-1}),$$

where $g_2(B)$ and $g_3(B)$ are of order $O(m^{-1})$, but $g_1(B)$ is of order $O(1)$. This leads to the following second-order unbiased (or nearly unbiased) estimator of $MSE(\hat{\theta}_i^{EB})$:

$$V_i^{PR} = g_1(\tilde{B}) + g_2(\tilde{B}) + g_3(\tilde{B})$$

$$= (1 - \tilde{B}) + \frac{\tilde{B}}{m} + \frac{2\tilde{B}}{m},$$

(Note we do not need a bias-correction for $g_1(\tilde{B})$ for this simple case). We have the second-order unbiasedness property of $V_i^{PR}$

$$E[V_i^{PR}] = MSE(\hat{\theta}_i^{EB}) + o(m^{-1}),$$
\[ V_{i}^{BL} = g_1(\hat{B}) + g_2(\hat{B}) - E_\ast \left[ \{g_1(\hat{B}^*) + g_1(\hat{B}^*)\} - \{g_1(\hat{B}) + g_2(\hat{B})\} \right] \]

\[ + E_\ast \left[ \hat{\theta}_{i}^{EB}(\hat{B}^*) - \hat{\theta}_{i}^{EB}(\hat{B}) \right]^2 \]

\[ \approx g_1(\hat{B}) + g_2(\hat{B}) - \frac{1}{R} \sum_{r=1}^{R} \left[ \{g_1(\hat{B}^{(r)}) + g_1(\hat{B}^{(r)})\} - \{g_1(\hat{B}) + g_2(\hat{B})\} \right] \]

\[ + \frac{1}{R} \sum_{r=1}^{R} \left[ \hat{\theta}_{i}^{EB}(\hat{B}^{(r)}) - \hat{\theta}_{i}^{EB}(\hat{B}) \right]^2 \]

\[ \approx (1 - \tilde{B}) + \frac{\tilde{B}}{m} + \frac{2\tilde{B}^2}{m} (Y_i - \bar{Y})^2, \]

- \( V_{i}^{BL} \) and \( V_{i}^{M} \) are identical up to order \( O_p(m^{-1}) \).
- \( V_{i}^{BL} \) satisfies the second-order unbiasedness property.
Other references on Parametric Bootstrap

- Efron (1982)
- Meza (2003)
- Hall and Maiti (2006a, 2006b)
- Hall (2006)
- Jiang and Lahiri (2006)
- Chatterjee and Lahiri (2007)
- González-Manteiga et al. (2007)
- González-Manteiga et al. (2008)
- Efron (2012)
A Class of Area Level Models

For $i = 1, \ldots, m$,

Level 1: $Y_i = h_{1i}^{-1}(\tilde{Y}_i)|\theta_i \overset{ind}{\sim} N(\theta_i, D_i),$

Level 2: $\theta_i \overset{ind}{\sim} N(x_i^T \beta, A),$

where

- $\tilde{Y}_i$: direct estimator of small area parameter of interest $h_{1i}(\theta_i)$ (mean, total, proportion)
- $h_{1i}(Y_i)$ is a one-to-one measurable function of $Y_i$
- $D_i$ is the sampling variance of $Y_i$ usually approximated or/and estimated using GVF; for inference from such models $D_i$'s are assumed to be known.
- $x_i^T$: a $p \times 1$ column vector of known auxiliary variables.
- Two types of parameters: $h_{1i}(\theta_i)$ (high-dimensional or small area parameters) and $(\beta, A)$ low dimensional hyperparameters
A Few Examples of Transformation Used Practice

1. SAIPE State level model for poverty rate: \( Y_i = \tilde{Y}_i; D_i \) are estimated by a replication-based method.

   \[ Y_i = \arcsin(\sqrt{\tilde{Y}_i}), \text{ where } n_i \text{ is the sample size for area } i; D_i = \frac{1}{4n_i}. \]
   In Chilean poverty mapping, similar transformation is used with \( n_i \) representing effective sample size to incorporate complex sample design.

3. Baseball Data Analysis (Efron and Morris 1975):
   \[ Y_i = \sqrt{n_i} \arcsin(2\tilde{Y}_i - 1), \text{ where } n_i \text{ is the sample size for area } i; D_i = 1. \]

4. Per-capita income (Fay and Herriot 1979): \( Y_{1i} = \log(\tilde{Y}_i); D_i = 9/N_i, \text{ where } N_i \text{ is the population size.} \)
In practice, small area estimators are obtained using several steps. All or a subset of the following steps are generally used:

- **Step 1: The Best Prediction (Bayes) Estimator**
  
  Note that

  \[ \theta_i | Y_i; (\beta, A) \overset{ind}{\sim} N \left[ (1 - B_i) Y_i + B_i x_i^T \beta, (1 - B_i) D_i \right], \]

  where

  \[ B_i = \frac{D_i}{A + D_i}, \quad i = 1, \ldots, m. \]

  This leads to the best prediction (BP) (same as the Bayes) estimator given by

  \[ \tilde{\theta}_i = E[\theta_i | Y_i; (\beta, A)] = (1 - B_i) Y_i + B_i x_i^T \beta. \]
Step 2: Empirical Best Linear Unbiased or Empirical Bayes (EB)

\[ \hat{\theta}_{mi} = (1 - \hat{B}_i) Y_i + \hat{B}_i x_i^T \hat{\beta} \]

where

\[ B_i = \frac{D_i}{\hat{A} + D_i}, \quad i = 1, \ldots, m; \]

\[ \hat{\beta} \] and \[ \hat{A} \] are consistent estimators of \( \beta \) and \( A \), respectively.

Step 3: Winsorization or Limited Translation

\[ T_{m1i} = \begin{cases} L_{mi} & \text{if } \hat{\theta}_{mi} < L_{mi} < U_{mi}, \\ \hat{\theta}_{mi} & \text{if } L_{mi} < \hat{\theta}_{mi} < U_{mi}, \\ U_{mi} & \text{if } L_{mi} < U_{mi} < \hat{\theta}_{mi}. \end{cases} \]

\[ = \text{median}(L_{mi}, \hat{\theta}_{mi}, U_{mi}) \]
Step 4: Benchmarking:

\[ T_{m2i} = K_{mi} \left( \sum_{i=1}^{m} h_1i(T_{m1i}) \right)^{-1} h_1i(T_{m1i}), \]

where \( K_{mi} \) is a known constant, or

\[ T_{m2i} = K_{mi} - m^{-1} \left( \sum_{i=1}^{m} h_1i(T_{m1i}) \right) + h_1i(T_{m1i}). \]

In general,

\[ T_{m2i} = h_{m2i}(T_{m11}, \ldots, T_{m1n}, K_{mi}) \]

where \( h_{m2i} \) is a continuous, and hence measurable, function.
How do we estimate MSE of $T_{m2i}$?

- Apply parametric bootstrap method. This is quite straightforward – we imitate the above steps on data generated using $\hat{\beta}$ and $\hat{A}$.

- Mean squared error (MSE) of $T_{m2i}$ is estimated by $E_* [T_{m2i*} - h_{1i}(\theta_{i*})]^2$, where $E_*$ is expectation with respect to the parametric bootstrap distribution of $[T_{m2i*}, h_{1i}(\theta_{i*})]$.

- In practice, we use the Monte Carlo approximation to $E_* [T_{m2i*} - h_{1i}(\theta_{i*})]^2$:

$$\hat{\text{MSE}}_{B1, Direct} = \frac{1}{B} \sum_{b=1}^{B} [T_{m2i,b} - h_{1i}(\theta_{i,b})]^2,$$

where $T_{m2i,b}$ and $h_{1i}(\theta_{i,b})$ are based on the $b$th parametric bootstrap sample ($b = 1, \ldots, B$).

- Under regularity conditions,

$$E \left[ \hat{\text{MSE}}_{B1, Direct} \right] = \text{MSE} + O(m^{-1}) + O(B^{-1/2}).$$
Here is why parametric bootstrap works: the functions $h_{1i}$ and $h_{2i}$ present no challenges, either they are 1-1 or continuous. The Winsorization step is about computing a median of three random variables for each small area. So, we are looking to estimate the distribution of a smooth function of

$$Z_{mi} = (Y_1, \ldots, Y_m, \theta_1, \ldots, \theta_m, \ K_{mi}, L_{m1}, \ldots, L_{mm}, U_{m1}, \ldots, U_{mm}) \in \mathbb{R}^{4m+1},$$

whose distribution is $F_Z(\cdot; \beta, A)$. Our assumptions make sure that $F_Z$ is a smooth function of the parameters, and hence the parametric bootstrap approximation $F_Z(\cdot; \hat{\beta}, \hat{A})$ converges. There is some more details here that we are skipping.
General MSPE Estimation: Double bootstrap

- Estimate parameters \( \xi = (\beta, A) \) with \( \hat{\xi} = (\hat{\beta}, \hat{A}) \).
- For \( b = 1, \ldots, B \), generate parametric bootstrap results.
- Within each first-layer parametric bootstrap step \( b = 1, \ldots, B \), implement a further round of bootstrap steps based on \( \hat{\xi}_b \) to get \( Y_{ib\tilde{b}}'s \), \( \theta_{ib\tilde{b}}'s \), \( T_{ib\tilde{b}}'s \) etc, for \( \tilde{b} = 1, \ldots, \tilde{B} \).
- For each \( b = 1, \ldots, B \) and for each \( \tilde{b} = 1, \ldots, \tilde{B} \) within each \( b \), obtain \((\theta_{ib\tilde{b}} - T_{ib\tilde{b}})^2\), and correct the bias in \( \widehat{MSPE}_{B1.Direct} \) using these to get \( \widehat{MSPE}_{B2.Direct} \).

\[
E \left[ \widehat{MSPE}_{B2.Direct} \right] = MSPE + O(n^{-3/2}) + O(B^{-1/2}) + O(\tilde{B}^{-1/2}).
\]

- There is no unique, good way of combining the \( \widehat{MSPE} \)'s. Several choices available (see Hall (1988, 1992), others).
- This is intense computation, \( O(B\tilde{B}) \). Two seconds of PB is 1.5 days of DPB!! \((n = 15 \rightarrow B = \tilde{B} \approx 3000 \rightarrow B\tilde{B} \approx 10^7)\)
A Monte Carlo Simulation

- For the simulation, we consider the following area level model:

\[ Y_i = h^{-1}_i(\bar{Y}_i) = \mu + v_i + e_i, \quad i = 1, \ldots, m, \]

where \(\{v_i\}\) and \(\{e_i\}\) are independent with \(e_i \sim (0, D_i)\) and \(v_i \sim (0, A)\) denoting a distribution with mean \(a\) and variance \(b\).

- Consider 5 groups of small areas, each with 3 small areas with the same \(D_i\). That is \(m = 15\).

- \(D_i\) pattern: \((2, 0.6, 0.5, 0.4, 0.2)\). That is, each of the 3 small areas in the first group has \(D_i = 2\), each of the 3 small areas in the second group has \(D_i = 0.6\), and so on.

- To generate \(\{(Y_i, \theta_i), \quad i = 1, \ldots, m\}\), we set unknown hyperparameters as \(A = 1\) and \(\mu = 0\).
We generate $R = 1000$ independent sets of $\{(Y_i, \theta_i), i = 1, \cdots, m\}$ for each of the following two cases:

- **Case 1**: No transformation, that is, $Y_i = \tilde{Y}_i$; both $\{e_i\}$ and $\{v_i\}$ are independently generated from normal distributions.

- **Case 2**: Exactly like in Case 1 except that $v_i$ are generated from a shifted exponential or double exponential distribution.

**Target parameters**: $\theta_i$ for both cases 1 and 2
For Cases 1 and 2, we consider the following two estimators

(i) \( T_{1i}, \text{ EB} \);

(ii) Winsorized and benchmarked version of EB, say \( T_{2i} \). That is, we first winsorise \( T_{1i} \) so that winsorised \( T_{1i} \), say \( T_{1i}^{\text{Win}} \), does not deviate from the direct \( Y_i \) by more than \( \sqrt{D_i} \) and then benchmark \( T_{1i}^{\text{Win}} \) so that the final estimates \( T_{2i} \) add up to \( \sum_{j=1}^{m} Y_j \).
For cases 1 and 2, we consider the following MSE estimators of $T_1$:

- Naive;
- PR Taylor series;
- BL bias-corrected parametric bootstrap;
- PG parametric bootstrap;
- Our proposed single-stage parametric bootstrap.

For $T_2$, we take the same.
Monte Carlo Simulation: Cases 1 and 2

<table>
<thead>
<tr>
<th>$D$</th>
<th>2.000</th>
<th>0.60</th>
<th>0.50</th>
<th>0.40</th>
<th>0.200</th>
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<tbody>
<tr>
<td>$T_1$</td>
<td>78.16</td>
<td>43.51</td>
<td>38.81</td>
<td>33.54</td>
<td>20.24</td>
</tr>
<tr>
<td>$T_2$</td>
<td>39.071</td>
<td>22.13</td>
<td>19.83</td>
<td>17.23</td>
<td>10.665</td>
</tr>
</tbody>
</table>

**Table:** True MSE values for EBLUP ($T_1$) and Winsorized, benchmarked EBLUP ($T_2$); the Prasad-Rao parameter estimators used.
Monte Carlo Simulation: Cases 1 and 2

Comparison of different estimators of MSE of $T_1$

Percent Relative Bias = $100 \times \frac{E(\text{MSE}) - \text{MSE}}{\text{MSE}}$

<table>
<thead>
<tr>
<th>$D$</th>
<th>2.000</th>
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<th>0.40</th>
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</tr>
</thead>
<tbody>
<tr>
<td>True MSE</td>
<td>78.16</td>
<td>43.51</td>
<td>38.81</td>
<td>33.54</td>
<td>20.24</td>
</tr>
<tr>
<td>Naive</td>
<td>-18.16</td>
<td>-21.71</td>
<td>-22.22</td>
<td>-22.96</td>
<td>-25.08</td>
</tr>
<tr>
<td>PR</td>
<td>4.35</td>
<td>8.34</td>
<td>9.73</td>
<td>12.00</td>
<td>29.71</td>
</tr>
<tr>
<td></td>
<td>(2.744)</td>
<td>(3.396)</td>
<td>(2.812)</td>
<td>(2.893)</td>
<td>(2.482)</td>
</tr>
<tr>
<td>B1.PG11</td>
<td>0.368</td>
<td>-2.371</td>
<td>-2.525</td>
<td>-2.763</td>
<td>-2.05</td>
</tr>
<tr>
<td>BL</td>
<td>1.965</td>
<td>-1.587</td>
<td>-1.932</td>
<td>-1.996</td>
<td>-1.47</td>
</tr>
<tr>
<td>B2</td>
<td>-0.921</td>
<td>-1.163</td>
<td>-3.48</td>
<td>-3.002</td>
<td>-0.03</td>
</tr>
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</table>

Table: True MSE values and percentage relative biases for naive, Prasad-Rao Taylor series (PR), parametric bootstrap with no bias correction (B1.Direct), Pfeffermann-Glickmann bias corrected parametric bootstrap (B1.PG11), Butar-Lahiri bias-corrected parametric bootstrap (BL) MSE estimators. B2 is double bootstrap. Red = < 1%, Magenta = < 3%, Fuschia = < 10%; numbers in the parentheses are for Case 2.
Monte carlo Simulation: Cases 1 and 2

Comparison of different estimators of MSE of $T_2$

<table>
<thead>
<tr>
<th>$D$</th>
<th>2.000</th>
<th>0.60</th>
<th>0.50</th>
<th>0.40</th>
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<tbody>
<tr>
<td>True MSE</td>
<td>39.071</td>
<td>22.13</td>
<td>19.83</td>
<td>17.23</td>
<td>10.665</td>
</tr>
<tr>
<td>Naive</td>
<td>60.5</td>
<td>55.2</td>
<td>54.0</td>
<td>52.3</td>
<td>45.4</td>
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<tr>
<td>PR</td>
<td>106.0</td>
<td>111.3</td>
<td>111.5</td>
<td>111.7</td>
<td>115.2</td>
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<tr>
<td></td>
<td>(4.092)</td>
<td>(2.532)</td>
<td>(2.02)</td>
<td>(1.372)</td>
<td>(1.142)</td>
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<tr>
<td>B1.PG11</td>
<td>99.173</td>
<td>96.31</td>
<td>95.74</td>
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<td>92.599</td>
</tr>
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<td>23.21</td>
<td>24.50</td>
<td>30.620</td>
</tr>
<tr>
<td>B2</td>
<td>(-4.018)</td>
<td>(-1.92)</td>
<td>(-3.003)</td>
<td>(-2.996)</td>
<td>(0.938)</td>
</tr>
</tbody>
</table>

Table: True MSE values and percentage relative biases for several MSE estimators. Fuschia= < 10%; numbers in the parentheses are for Case 2.
Thank you