3. Correlation analysis

- Correlation and Covariance functions
- Analysis on LTI systems

Correlation and Covariance

Suppose X, Y are random variables with means μ_x and μ_y resp.

Cross Correlation:

$$R_{xy} = \mathbf{E}[XY^*]$$

Cross Covariance:

$$C_{xy} = \mathbf{E}\left[(X - \mu_x)(Y - \mu_y)^*\right]$$

Autocorrelation:

$$R = \mathbf{E}[XX^*]$$

Autocovariance:

$$C = \mathbf{E}\left[(X - \mu_x)(X - \mu_x)^*\right]$$

correlation = covariance when considering zero mean

Correlation and Covariance functions

Suppose $x(t), y(t)$ are random processes

Cross Correlation: $R_{xy}(t_1, t_2) = \mathbf{E}\, x(t_1)y(t_2)^*$

Cross Covariance:

$$C_{xy}(t_1, t_2) = \mathbf{E}\left[(x(t_1) - \mu_x(t_1))(y(t_2) - \mu_y(t_2))^*\right]$$

where $\mu_x(t) = \mathbf{E}\, x(t)$ and $\mu_y(t) = \mathbf{E}\, y(t)$

Autocorrelation: $R(t_1, t_2) = \mathbf{E}\, x(t_1)x(t_2)^*$

Autocovariance:

$$C(t_1, t_2) = \mathbf{E}\left[(x(t_1) - \mu(t_1))(x(t_2) - \mu(t_2))^*\right]$$

Wide-sense stationary process

Strictly stationary process: the joint distribution is invariant with time

Weakly (wide-sense) stationary process:

1. $\mathbf{E}\, x(t) =$ constant
2. $R(t_1, t_2) = R(t_1 - t_2)$

With wide-sense stationary assumption, the correlation function is given by

$$R_{xy}(\tau) = \mathbf{E}\, x(t+\tau)y(t)^*$$

and the covariance function is simplified to

$$C_{xy}(\tau) = \mathbf{E}\, x(t+\tau)y(t)^* - \mu_x \mu_y^*$$

Example

Determine the mean and the autocorrelation of a random process

$$x(t) = A\cos(\omega t + \phi)$$

where the random variables A and ϕ are independent and ϕ is uniform on $(-\pi, \pi)$

Since A and ϕ are independent, the mean is given by

$$\mathbf{E}\, x(t) = \mathbf{E}[A]\, \mathbf{E}[\cos(\omega t + \phi)]$$

Using the uniform distribution in ϕ, the last term is

$$\mathbf{E}\cos(\omega t + \phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \phi) d\phi = 0$$

Therefore, $\mathbf{E}\, x(t) = 0$

Example (cont.)

Using trigonometric identities, the autocorrelation is determined by

$$\mathbf{E}\, x(t_1)x(t_2) = \frac{1}{2}\,\mathbf{E}\, A^2\, \mathbf{E}[\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\phi)]$$

Since

$$\mathbf{E}[\cos(\omega t_1 + \omega t_2 + 2\phi)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 + 2\phi) d\phi = 0$$

Therefore,

$$R(t_1, t_2) = (1/2)\,\mathbf{E}[A^2]\cos\omega(t_1 - t_2)$$

We conclude that the random process in this example is wide-sense stationary

Connection with spectral density

Wiener-Khinchin theorem:

If a process is wide-sense stationary, the autocorrelation function and the power spectral density form a Fourier transform pair:

$$S(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega\tau} R(\tau) d\tau \qquad \text{Continuous}$$

$$S(\omega) = \sum_{k=-\infty}^{k=\infty} R(k) e^{-\mathrm{i}\omega k} \qquad \text{Discrete}$$

Therfore, the autocorrelation function at $\tau = 0$ indicates the average power:

$$R(0) = \mathbf{E}[x(t)x(t)^*] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega$$

(similarly, use discrete inverse Fourier transform for discrete systems)

Properties of autocorrelation functions

- $R(-t) = R(t)^*$ (if the process is real and scalar, then $R(-t) = R(t)$
- Non-negativity: that is for any $a_i, a_j \in \mathbf{C}^n$, with $i, j = 1, \ldots N$, we have

$$\sum_i^N \sum_j^N a_i^* R(i-j) a_j \geq 0,$$

which follows from

$$\sum_i^N \sum_j^N a_i^* R(i-j) a_j = \sum_i^N \sum_j^N \mathbf{E}[a_i^* x(i) x(j)^* a_j] = \mathbf{E}\left[\left(\sum_i^N a_i^* x(i)\right)^2\right] \geq 0.$$

Correlation analysis

Consider a discrete LTI system with a disturbance $v(t)$

$$y(t) = \sum_{k=0}^{\infty} h(k)u(t-k) + v(t)$$

Assume u, v have zero mean and $\mathbf{E}\, u(t)v(s)^* = 0, \forall t, s$.

The correlation function is given by

$$R_{yu}(\tau) = \mathbf{E}\, y(t+\tau)u(t)^* = \sum_{k=0}^{\infty} h(k)R_u(\tau - k)$$

If $u(t)$ is *white noise* ($R_u(\tau) = 0, \tau \neq 0$), it is simplified to

$$R_{yu}(k) = h(k)R_u(0)$$

Correlation analysis (cont.)

Use finite approximation of $R_{yu}(k)$ and $R_u(0)$ to solve for $h(k)$

$$\hat{R}_{yu}(\tau) = \frac{1}{N}\sum_{t=1}^{N-\tau} y(t+\tau)u(t)^*, \quad \tau = 0, 1, 2, \ldots$$

$$\hat{R}_{uu}(\tau) = \frac{1}{N}\sum_{t=1}^{N-\tau} u(t+\tau)u(t)^*, \quad \tau = 0, 1, 2, \ldots$$

When $u(t)$ is not exactly white

- filter both inputs and outputs that makes the input as white as possible
- truncate the impulse response at a certain order

FIR model

Assume that

$$h(k) = 0, \quad k > M$$

This is called *a finite impulse respose (FIR)* or a *truncated weighting function*

The correlation equation becomes

$$R_{yu}(\tau) = \sum_{k=0}^{M} h(k) R_u(\tau - k)$$

Writing out this equation for $\tau = 0, 1, \ldots, M$ gives a linear equation:

$$\begin{bmatrix} R_{yu}^*(0) \\ R_{yu}^*(1) \\ \vdots \\ R_{yu}^*(M) \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(1) & \cdots & R_u(M) \\ R_u(-1) & R_u(0) & \cdots & R_u(M-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_u(-M) & R_u(-M+1) & \cdots & R_u(0) \end{bmatrix} \begin{bmatrix} h^*(0) \\ h^*(1) \\ \vdots \\ h^*(M) \end{bmatrix}$$

Example with white noise input

Consider a scalar system

$$x(t) + ax(t-1) = bu(t-1), \quad |a| < 1$$
$$y(t) = x(t) + v(t)$$

with $a = 0.5, b = 5$

Assume that $u(t)$ and $v(t)$ are independent white noise with variances $\sigma_u^2 = \sigma_v^2 = 0.1$

The transfer function is

$$H(z) = \frac{bz^{-1}}{1 + az^{-1}} = b(z^{-1} - az^{-2} + a^2 z^{-3} - a^3 z^{-4} + \dots)$$

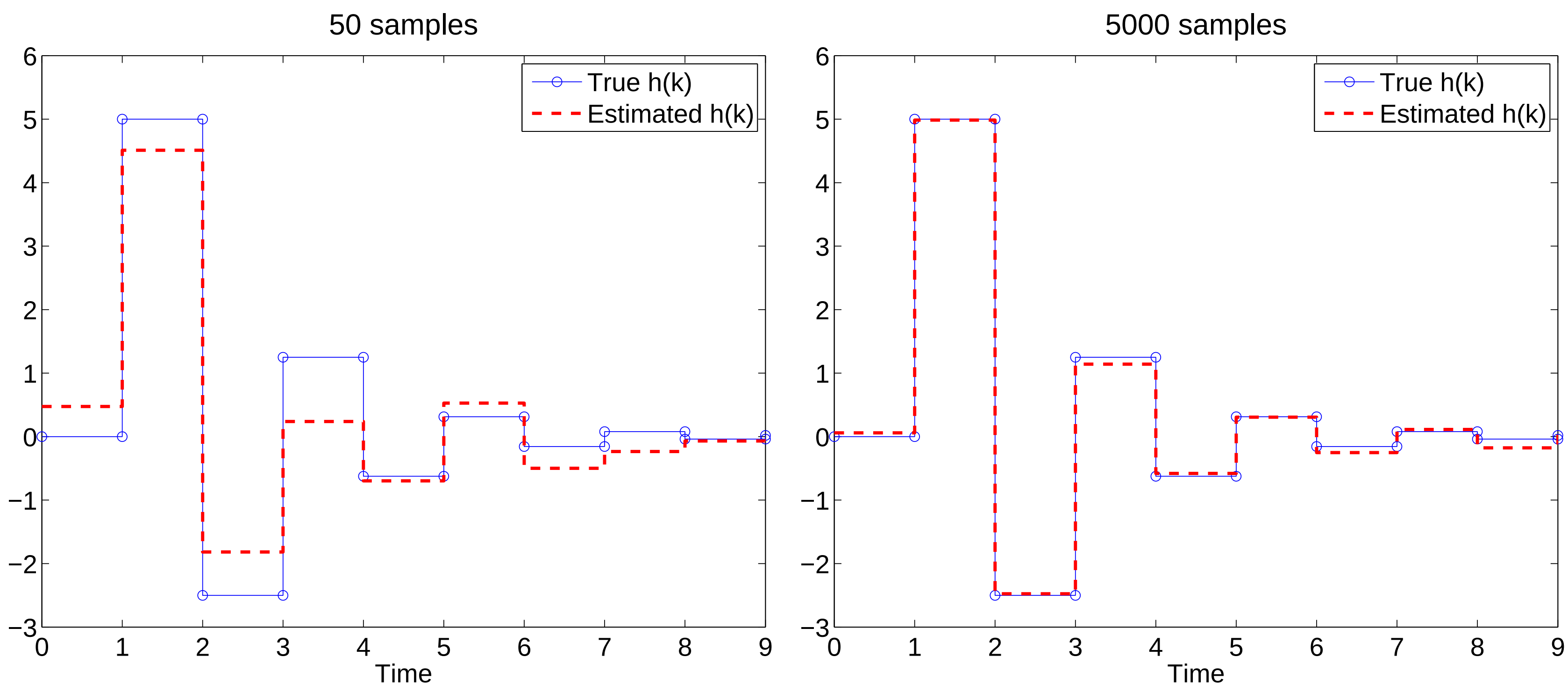
The impulse response is therefore given by

$$h(0) = 0, \quad h(k) = b(-a)^{k-1}, k \geq 1$$

Example with white noise input

The estimate of the impulse response is

$$\hat{h}(k) = \hat{R}_{yu}(k)/\hat{\sigma}_u^2$$



$\hat{\sigma}_u^2 = 0.093$ $\hat{\sigma}_u^2 = 0.099$

References

Chapter 6 in
L. Ljung, *System Identification: Theory for the User*, Prentice Hall, Second edition, 1999

Chapter 3 in
T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989