4. Fourier analysis

- Empirical transfer function estimate
- Discrete Fourier Transform of a finite-length signal
- Properties of ETFE

Empirical transfer function estimate

Consider a linear system with the representation

$$Y(\omega) = G(\omega)U(\omega)$$

We extend the frequency analaysis to the case of multifrequency inputs

An estimate of the transfer function:

$$\hat{G}(\omega) = Y(\omega)U(\omega)^{-1}$$

is proposed and it is called the *empirical transfer-function estimate (ETFE)*

Estimates of $Y(\omega), U(\omega)$ are given by the Fourier transform of the finite sequences:

$$Y_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} y(t)e^{-\mathrm{i}\omega t}, \quad U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t)e^{-\mathrm{i}\omega t}$$

DFT of a finite-length signal

The DFT of the length-N time-domain sequence $x[n]$ is defined by

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-\mathrm{i}2\pi kn/N}, \quad 0 \le k \le N-1$$

The sequences $X[k]$ are, in general, complex numbers even $x[n]$ are real

The inverse DFT is given by

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{\mathrm{i}2\pi kn/N}, \quad 0 \le n \le N-1$$

Matrix relation of DFT

Define $W = e^{-i2\pi/N}$, we can write the DFT in a matrix form as

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W^1 & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

or

$$\mathbf{X} = \mathbf{D}\mathbf{x},$$

where $\mathbf{D}$ is called the *DFT matrix*

The inverse DFT is given by $\mathbf{x} = \mathbf{D}^{-1}\mathbf{X}$ and using the fact that

$$\mathbf{D}^{-1} = \mathbf{D}^*$$

($\mathbf{D}$ is called an *orthogonal matrix*, i.e., $\mathbf{D}^*\mathbf{D} = I$)

Orthogonality of DFT matrix

Define ϕ_k the k-th column of DFT matrix:

$$\phi_k = (1/\sqrt{N}) \begin{bmatrix} 1 & W^k & W^{2k} & \cdots & W^{k(N-1)} \end{bmatrix}^T,$$

or equivalently

$$\phi_k = (1/\sqrt{N}) \begin{bmatrix} 1 & e^{-\mathrm{i}2\pi k/N} & e^{-\mathrm{i}2\pi k\cdot 2/N} & \cdots & e^{-\mathrm{i}2\pi k(N-1)/N} \end{bmatrix}^T$$

Let $u = e^{\mathrm{i}2\pi(k-l)/N}$ and use a result of the sum of geometric series:

$$\langle \phi_l, \phi_k \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{\mathrm{i}2\pi(k-l)n/N} = \frac{1 - e^{\mathrm{i}2\pi(k-l)}}{1 - e^{\mathrm{i}2\pi(k-l)/N}}$$

We can show that

$$\langle \phi_l, \phi_k \rangle = \begin{cases} 1, & \text{for } k = l + rN, \quad r = 0, 1, 2, \ldots \\ 0, & \text{for } k \neq l \end{cases}$$

Frequency sampling of the Fourier transform

The Fourier transform of the length-N sequence $x[n]$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-\mathrm{i}\omega n} = \sum_{n=0}^{N-1} x[n] e^{-\mathrm{i}\omega n}$$

If we uniformly sampling $X(\omega)$ on the ω-axis between $[0, 2\pi)$ by

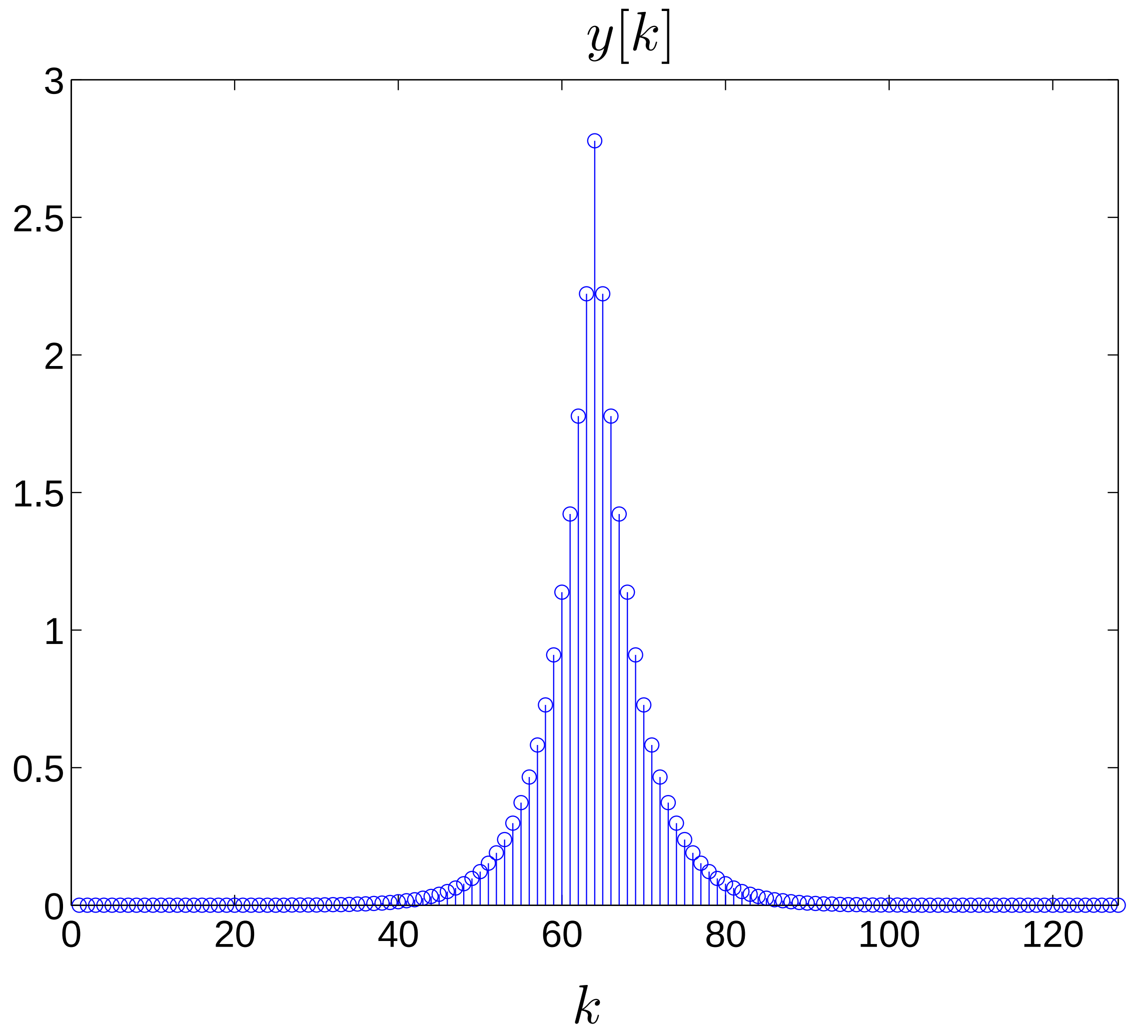
$$\omega_k = 2\pi k/N, \quad 0 \leq k \leq N-1,$$

then we have

$$X(\omega) \,|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-\mathrm{i}2\pi kn/N}, \quad 0 \leq k \leq N-1$$

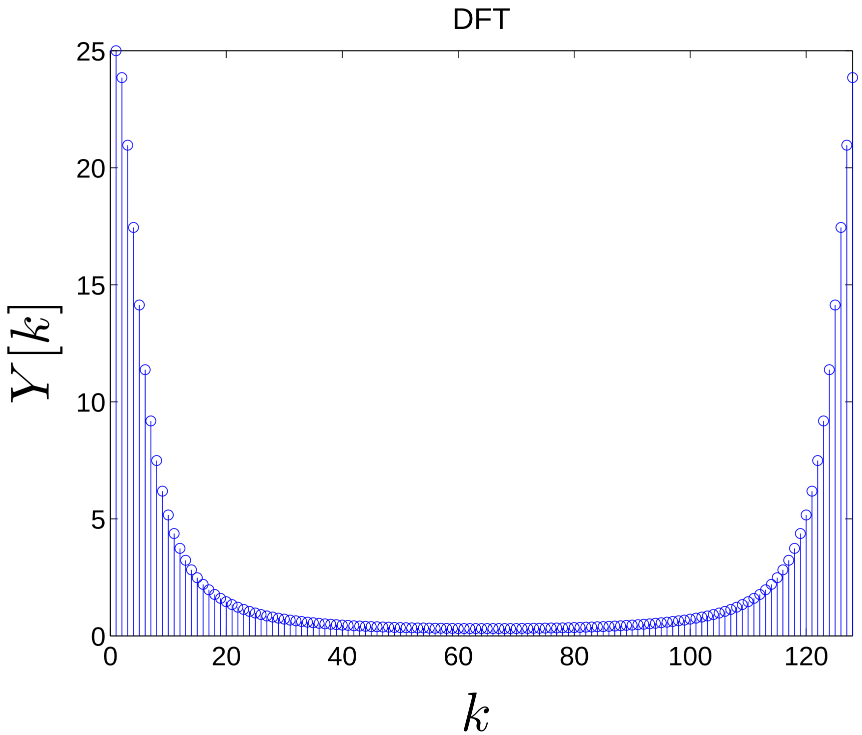
Frequency samples of $X(\omega)$ of length-N sequence $x[n]$ at N equally spaced frequencies is *precisely* the N-point DFT $X[k]$

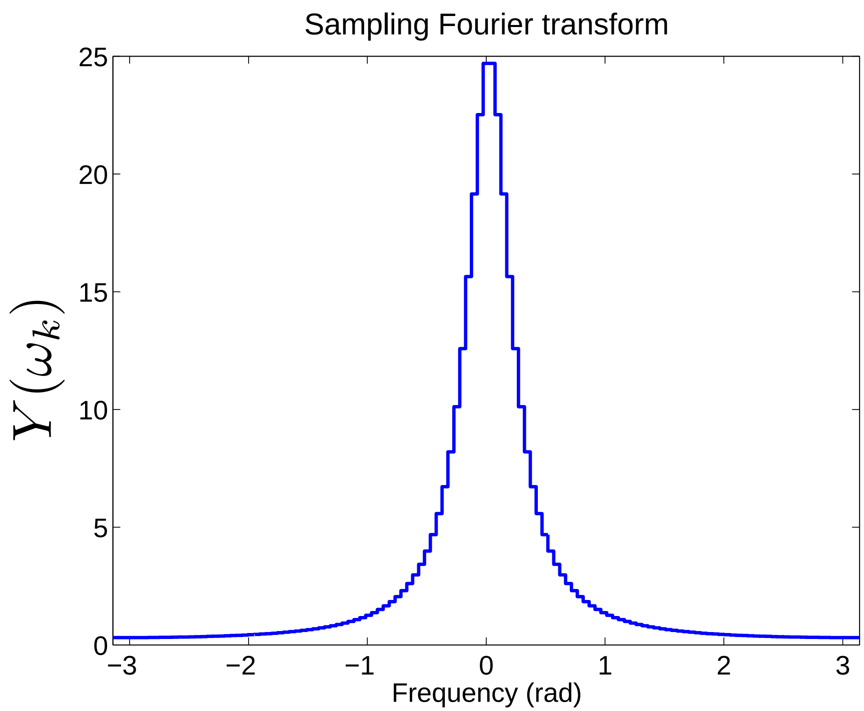
Computing DFT on MATLAB



- $y[k] = a^{|k|-64}/(1-a^2)$, with $a = 0.8$ (an autocorrelation sequence)

Computing DFT on MATLAB





- Use `fft` command
- $y[k]$ is symmetric about 0, and so is $Y[k]$
- Compare with $Y(\omega) = 1/(1 + a^2 - 2a\cos\omega)$

Transformation of DFT

Let $y(t)$ and $u(t)$ are related by a strictly linear SISO system:

$$y(t) = G(q)u(t)$$

where q is the forward shift operator and $G(q)$ is the transfer function

Assume that $|u(t)| \le C$ for all t and let

$$Y_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} y(t)e^{-\mathrm{i}\omega t}, \quad U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t)e^{-\mathrm{i}\omega t}$$

Then the DFTs of (windowed) $y(t)$ and $u(t)$ are related by

$$Y_N(\omega) = G(\omega)U_N(\omega) + R_N(\omega)$$

where

$$|R_N(\omega)| \le \frac{2KC}{\sqrt{N}} \qquad \text{(Ljung 1999, THM 2.1)}$$

Transformation of DFT

By definition

$$Y_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} y(t) e^{-\mathrm{i}\omega t} = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{k=1}^{\infty} g(k) u(t-k) e^{-\mathrm{i}\omega t}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} g(k) e^{-\mathrm{i}\omega k} \cdot \sum_{\tau=1-k}^{N-k} u(\tau) e^{-\mathrm{i}\omega \tau}$$

The last term on RHS is deviated from $U_N(\omega)$ by

$$|U_N(\omega) - \frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{N-k} u(\tau) e^{-\mathrm{i}\omega\tau}|$$

$$\leq |\frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{0} u(\tau) e^{-\mathrm{i}\omega\tau}| + |\frac{1}{\sqrt{N}} \sum_{\tau=N-k+1}^{N} u(\tau) e^{-\mathrm{i}\omega\tau}| \leq \frac{2Ck}{\sqrt{N}}$$

Transformation of DFT

Therefore,

$$
\begin{aligned}
|Y_N(\omega) - G(\omega)U_N(\omega)| &= \left| \sum_{k=1}^{\infty} g(k)e^{-\mathrm{i}\omega k} \left(\frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{N-k} u(\tau)e^{\mathrm{i}\omega\tau} - U_N(\omega) \right) \right| \\
&\leq \frac{2}{\sqrt{N}} \sum_{k=1}^{\infty} |kg(k)Ce^{-\mathrm{i}\omega k}|
\end{aligned}
$$

If we let $K = \sum_{k=1}^{\infty} k|g(k)|$, then the above inequality is bounded by

$$
|Y_N(\omega) - G(\omega)U_N(\omega)| \leq \frac{2KC}{\sqrt{N}}
$$

Note: If $u(t)$ is periodic with period N, then $R_N(\omega)$ is zero for $\omega = 2\pi k/N$

Properties of ETFE

Consider a linear model with disturbance

$$y(t) = G(q)u(t) + v(t)$$

From the previous page, we found that

$$\hat{G}(\omega) = G(\omega) + \frac{R_N(\omega)}{U_N(\omega)} + \frac{V_N(\omega)}{U_N(\omega)}$$

where $V_N(\omega)$ denotes the Fourier transform of the disturbance term

If we assume that $v(t)$ has zero mean, then

$$\mathbf{E}\,\hat{G}(\omega) = G(\omega) + \frac{R_N(\omega)}{U_N(\omega)}$$

The estimate has a bias term which decays as $1/\sqrt{N}$

Properties of ETFE

It can be shown that (Ljung 1999,§6.3)

$$\mathbf{E}[(\hat{G}(\omega) - G(\omega))(\hat{G}(\lambda) - G(\lambda))^*]$$
$$= \begin{cases} \dfrac{S_v(\omega) + \rho_N}{|U_N(\omega)|^2}, & \text{if} \quad \lambda = \omega \\ \dfrac{\rho_N}{U_N(\omega)U_N(-\lambda)}, & \text{if} \quad |\lambda - \omega| = \frac{2\pi k}{N}, k = 1, 2, \ldots \end{cases}$$

with $|\rho_N|$ is bounded by $1/N$ (up to a constant factor)

Conclusions

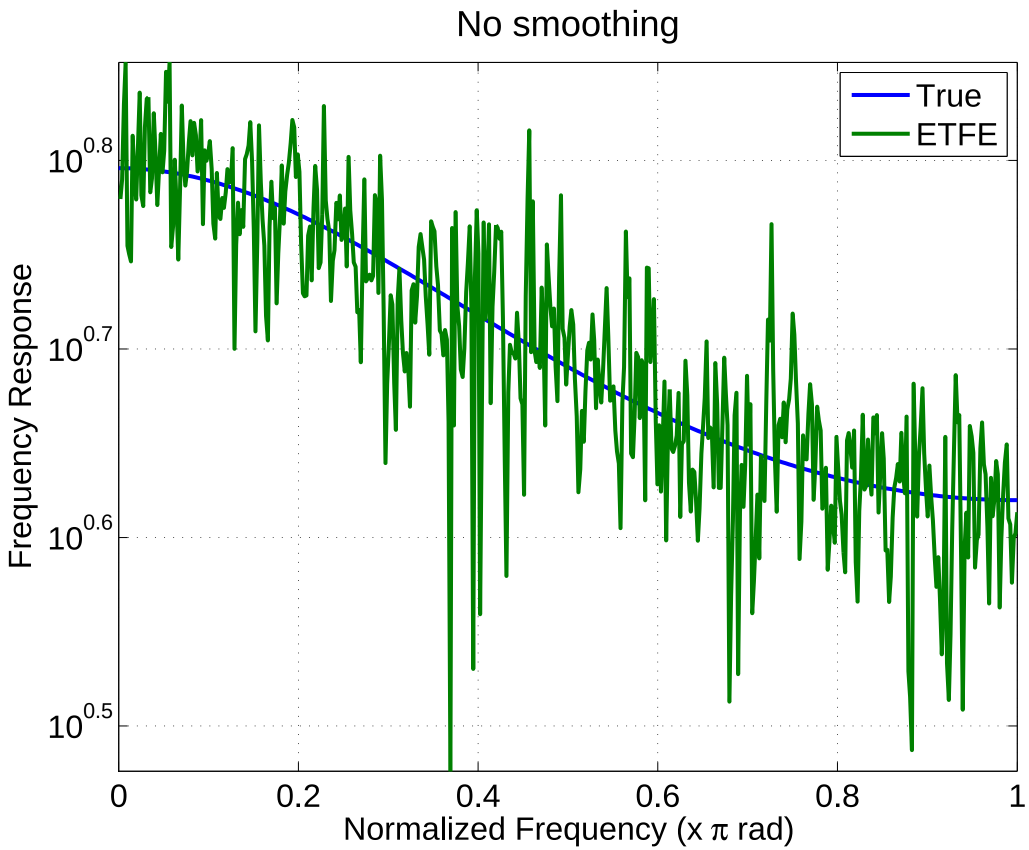
Periodic inputs

- $\hat{G}(\omega)$ is defined only for a fixed number of frequencies
- At these frequencies the ETFE is unbiased and its variance decays like $1/N$

Nonperiodic inputs

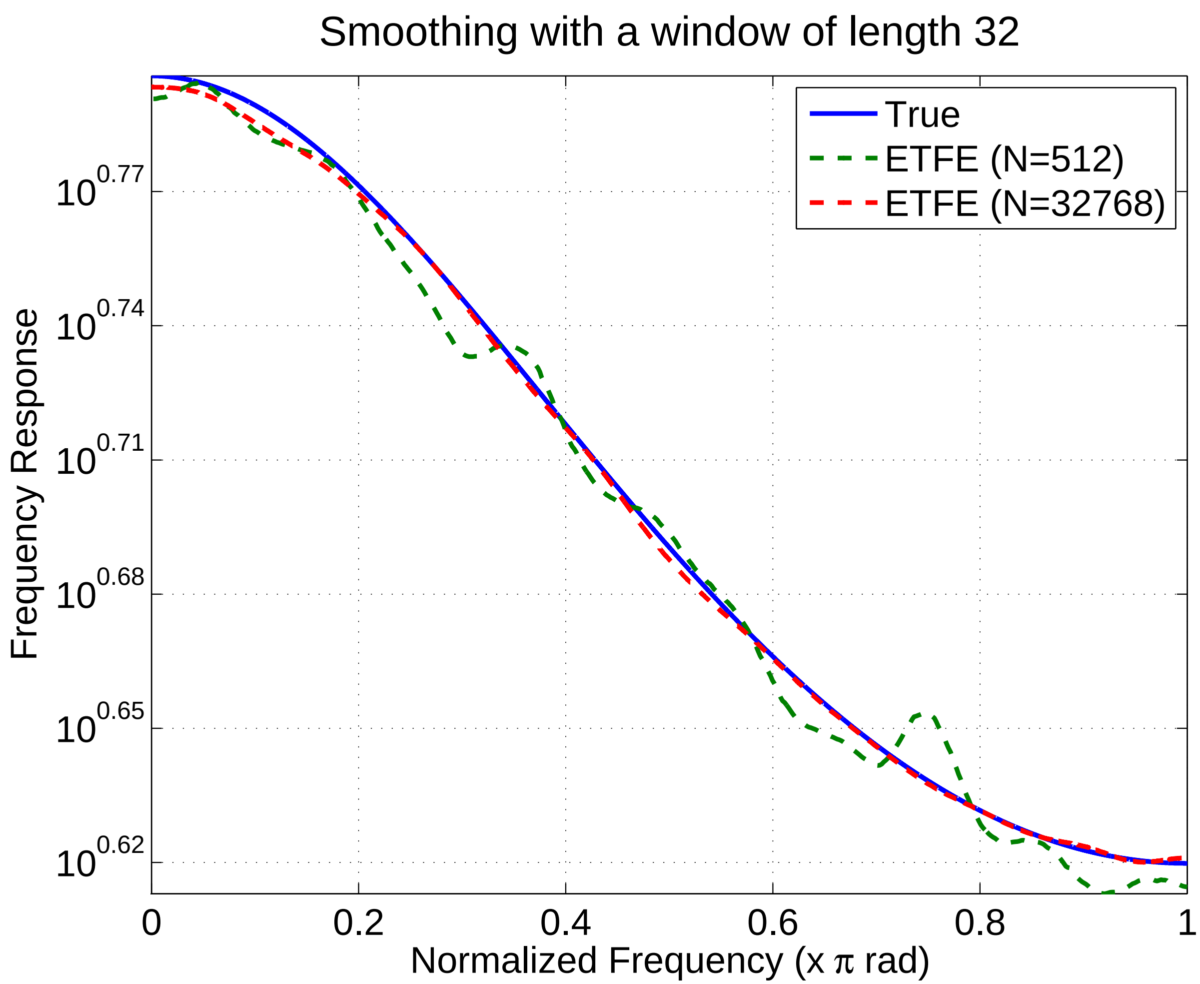
- $\hat{G}(\omega)$ is an asymptotically unbiased estimate of $G(\omega)$ at many frequencies
- The variance of $\hat{G}(\omega)$ does not decay with N but is given as the noise-to-signal ratio at the frequency in question as N increases
- This property makes the empirical estimate a crude estimate in most cases in practice

Example



- $G(z) = \frac{5}{z-0.2}$, white noise input with power 1, additive noise variance is 0.25
- Use `etfe` command in System Identification Toolbox

Example



Smoothing with a window of length 32
True
ETFE (N=512)
ETFE (N=32768)
Frequency Response
$10^{0.77}$
$10^{0.74}$
$10^{0.71}$
$10^{0.68}$
$10^{0.65}$
$10^{0.62}$
0
0.2
0.4
0.6
0.8
1
Normalized Frequency (x π rad)

References

Chapter 6 in
L. Ljung, *System Identification: Theory for the User*, Prentice Hall, Second edition, 1999

Chapter 5 in
S. K. Mitra, *Digital Signal Processing*, McGraw-Hill, International edition, 2006