6. Input signals

- Common input signals in system identification
  - step function
  - sum of sinusoids
  - ARMA sequences
  - Pseudo random binary sequence
- Spectral characteristics
- Persistent excitation

Step function

A step function is given by

$$u(t) = \begin{cases} 0, & t < 0 \\ u_0, & t \geq 0 \end{cases}$$

where the amplitude u_0 is arbitrarily chosen

The step response can be related to rise time, overshoots, static gain, etc.

It is useful for systems with a large signal-to-noise ratio

Sum of sinusoids

The input signal $u(t)$ is given by

$$u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$$

where the angular frequencies $\{\omega_k\}$ are distinct,

$$0 \leq \omega_1 < \omega_2 < \ldots < \omega_m \leq \pi$$

and the amplitudes and phases a_k, ϕ_k are chosen by the user

Characterization of sinusoids

Let S_N be the average of a sinusoid over N points

$$S_N = \frac{1}{N}\sum_{t=1}^{N} a\sin(\omega t + \phi)$$

Let μ be the mean of the sinusoidal function

$$\mu = \lim_{N\to\infty} S_N = \begin{cases} a\sin\phi, & \omega = 2n\pi, n = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$ has zero mean if $\omega_1 > 0$

Otherwise, we can always subtract the mean from $u(t)$

WLOG, assume zero mean for $u(t)$

Spectrum of sinusoidal inputs

The autocorrelation function can be computed by

$$\begin{aligned} R(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t+\tau)u(t) \\ &= \sum_{k=1}^{m} C_k \cos(\omega_k \tau) \end{aligned}$$

with $C_k = a_k^2/2$ for $k = 1, 2, \ldots, m$

If $\omega_m = \pi$, the coefficient C_m should be modified by

$$C_m = a_m^2 \sin^2 \phi_m$$

Therefore, the spectrum is

$$S(\omega) = \sum_{k=1}^{m} (C_k/2) \left[\delta(\omega - \omega_k) + \delta(\omega + \omega_k)\right]$$

Autoregressive Moving Average sequence

Let $e(t)$ be a pseudorandom sequence similar to white noise in the sense that

$$\frac{1}{N}\sum_{t=1}^{N} e(t)e(t+\tau) \rightarrow 0, \quad \text{as } N \rightarrow \infty$$

A general input $u(t)$ can be obtained by linear filtering

$$u(t) + c_1 u(t-1) + \ldots c_p u(t-p) = e(t) + d_1 e(t-1) + \ldots + d_q e(t-p)$$

- u(t) is called *ARMA (autoregressive moving average)* process
- When all $c_i = 0$ it is called *MA (moving average)* process
- When all $d_i = 0$ it is called *AR (autoregressive)* process
- The user gets to choose c_i, d_i and the random generator of $e(t)$

ARMA sequence (cont.)

The transfer function from $e(t)$ to $u(t)$ can be written as

$$U(z) = \frac{D(z)}{C(z)} E(z)$$

where

$$C(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + \dots c_p z^{-p}$$
$$D(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \dots d_q z^{-q}$$

- The distribution of $e(t)$ is often chosen to be Gaussian
- c_i, d_i are chosen such that $C(z), D(z)$ have zeros outside the unit circle
- Different choices of c_i, d_i lead to inputs with various spectral characteristics

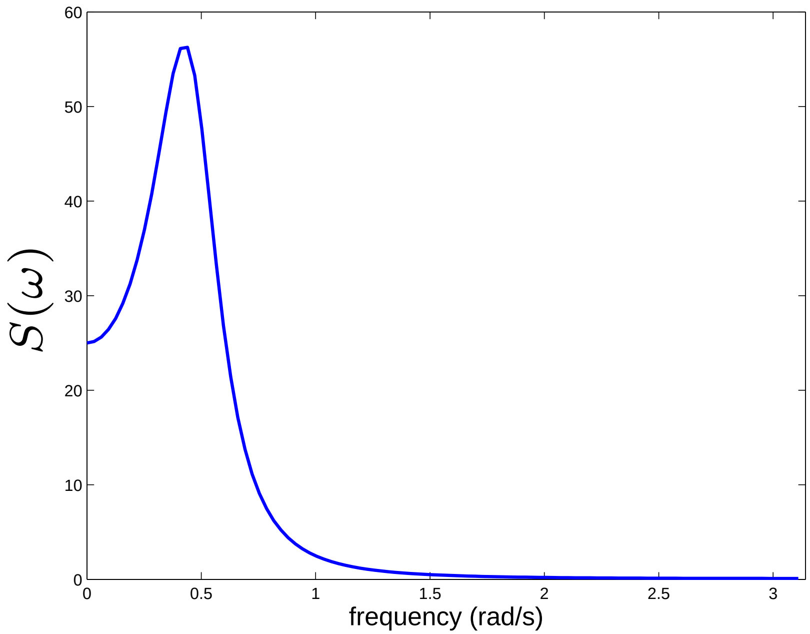
Spectrum of ARMA process

Let $e(t)$ be a white noise with variance λ^2

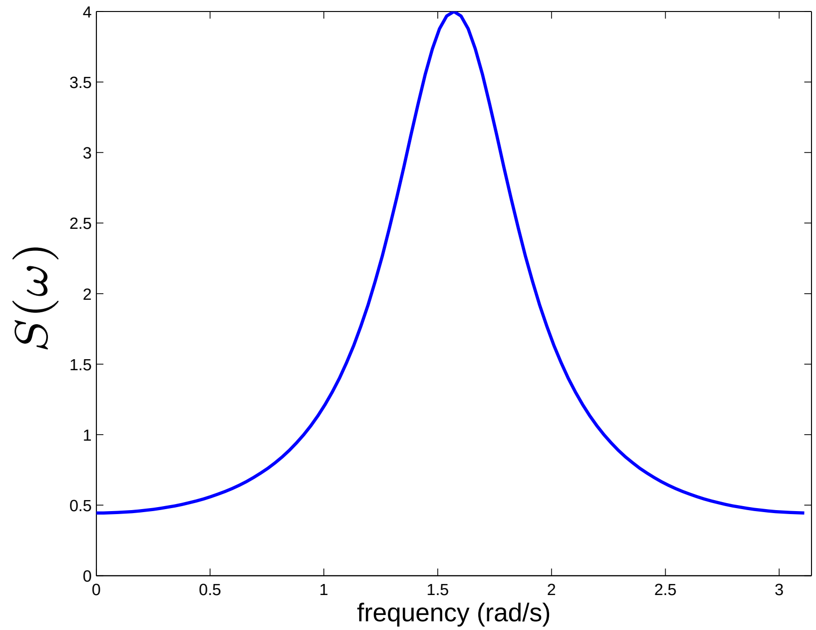
The spectral density of ARMA process is

$$S(\omega) = \lambda^2 \left| \frac{D(\omega)}{C(\omega)} \right|^2$$

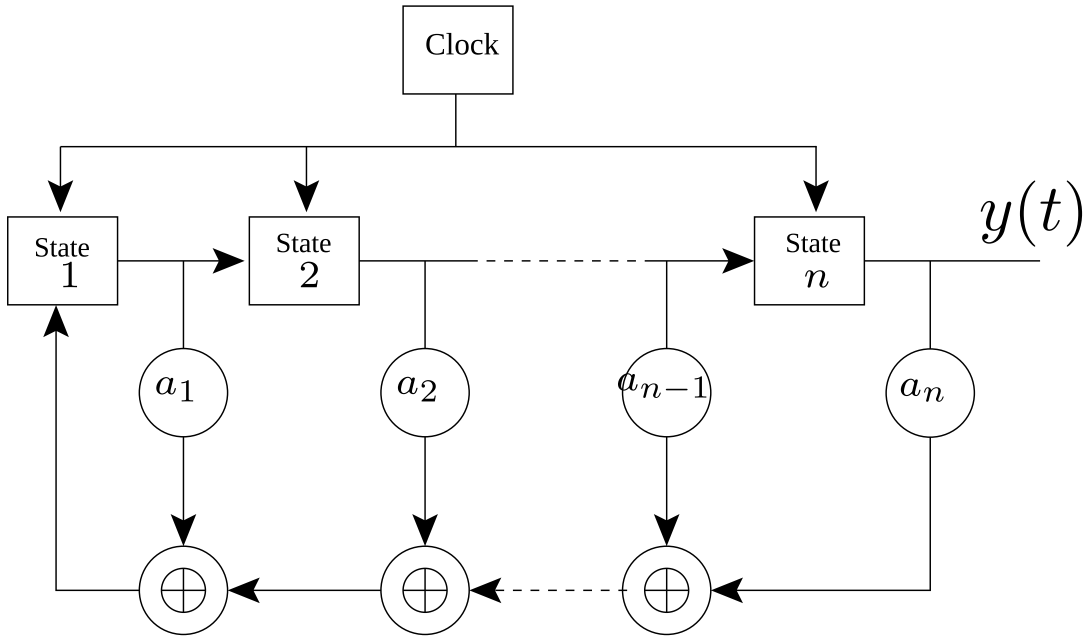
$\frac{D(z)}{C(z)} = 1/(z^2 - 1.5z + 0.7)$



$\frac{D(z)}{C(z)} = 1/(z^2 + 0.5)$



Pseudorandom binary sequence (PRBS)



$$x(t+1) = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} x(t)$$

PRBS (cont.)

- Every initial state is allowed except the all-zero state
- The feedback coefficients $a_1, a_2, \ldots, a_n$ are either 0 or 1
- All additions is modulo-two operation
- The sequences are two-state signals (binary)
- There are possible $2^n - 1$ different state vectors x (all-zero state is excluded)
- A PRBS of period equal to $M = 2^n - 1$ is called *a maximum length PRBS* (ML PRBS)
- For *maximum length PRBS*, its characteristic resembles white random noise (pseudorandom)

Influence of the feedback path on the period

Let $n = 3$ and initialize x with $x(0) = (1, 0, 0)$

- With $a = (1, 1, 0)$, the state vectors $x(k), k = 1, 2, \ldots$ are

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix}$$

  The sequence has period equal to 3

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  The sequence has period equal to 7 (the maximum period, $2^3 - 1$)

Maximum length PRBS

Denote q^{-1} the unit delay operator and let

$$A(q^{-1}) = 1 \oplus a_1 q^{-1} \oplus a_2 q^{-2} \oplus \ldots \oplus a_n q^{-n}$$

The PRBS $y(t)$ satisfies the homogeneous equation:

$$A(q^{-1}) y(t) = 0$$

This equation has only solutions of period $M = 2^n - 1$ *if and only if*

1. The binary polynomial $A(q^{-1})$ is irreducible, $i.e.$, there do not exist any two polynomial $A_1(q^{-1})$ and $A_2(q^{-1})$ such that

$$A(q^{-1}) = A_1(q^{-1}) A_2(q^{-1})$$

2. $A(q^{-1})$ is a factor of $1 \oplus q^{-M}$ but is not a factor of $1 \oplus q^{-p}$ for any $p < M$

Generating Maximum length PRBS

Examples of polynomials $A(z)$ satisfying the previous two conditions

n	$A(z)$	
3	$1 \oplus z \oplus z^3$	$1 \oplus z^2 \oplus z^3$
4	$1 \oplus z \oplus z^4$	$1 \oplus z^3 \oplus z^4$
5	$1 \oplus z^2 \oplus z^5$	$1 \oplus z^3 \oplus z^5$
6	$1 \oplus z \oplus z^6$	$1 \oplus z^5 \oplus z^6$
7	$1 \oplus z \oplus z^7$	$1 \oplus z^3 \oplus z^7$
8	$1 \oplus z \oplus z^2 \oplus z^7 \oplus z^8$	$1 \oplus z \oplus z^6 \oplus z^7 \oplus z^8$
9	$1 \oplus z^4 \oplus z^9$	$1 \oplus z^5 \oplus z^9$
10	$1 \oplus z^3 \oplus z^{10}$	$1 \oplus z^7 \oplus z^{10}$

Properties of maximum length PRBS

Let $y(t)$ be an ML PRBS of period $M = 2^n - 1$

- Within one period $y(t)$ contains $(M+1)/2 = 2^{n-1}$ ones and $(M-1)/2 = 2^{n-1} - 1$ zeros

- For $k = 1, 2, \ldots, M-1$,

$$y(t) \oplus y(t-k) = y(t-l)$$

  for some $l \in [1, M-1]$ that depends on k

Moreover, for any binary variables x, y,

$$xy = \frac{1}{2}\left(x + y - (x \oplus y)\right)$$

These properties will be used to compute the covariance function of maximum length PRBS

Covariance function of maximum length PRBS

The mean is given by counting the number of outcome 1 in $y(t)$:

$$m = \frac{1}{M}\sum_{t=1}^{M} y(t) = \frac{1}{M}\left(\frac{M+1}{2}\right) = \frac{1}{2} + \frac{1}{2M}$$

The mean is slightly greater than 0.5

Using $y^2(t) = y(t)$, we have the covariance function at lag zero as

$$C(0) = \frac{1}{M}\sum_{t=1}^{M} y^2(t) - m^2 = m - m^2 = \frac{M^2-1}{4M^2}$$

The variance is therefore slightly less than $1/4$

Covariance function of maximum length PRBS

For $\tau = 1, 2, \ldots,$

$$\begin{aligned} C(\tau) &= (1/M)\sum_{t=1}^{M} y(t+\tau)y(t) - m^2 \\ &= \frac{1}{2M}\sum_{t=1}^{M}[y(t+\tau) + y(t) - (y(t+\tau)\oplus y(t))] - m^2 \\ &= m - \frac{1}{2M}\sum_{t=1}^{M} y(t+\tau-l) - m^2 = m/2 - m^2 \\ &= -\frac{M+1}{4M^2} \end{aligned}$$

Asymptotic behavior of the covariance function of PRBS

Define $\tilde{y}(t) = -1 + 2y(t)$ so that its outcome is either -1 or 1

$$\begin{aligned} \tilde{m} &= -1 + 2m = 1/M \approx 0 \\ \tilde{C}(0) &= 4C(0) = 1 - 1/M^2 \approx 1 \\ \tilde{C}(\tau) &= 4C(\tau) = -1/M - 1/M^2 \approx -1/M, \quad \tau = 1, 2, \ldots, M-1 \end{aligned}$$

When M is large, the covariance function of PRBS has similar properties to a white noise

However, their spectral density matrices can be drastically different

Spectral density of PRBS

The output of PRBS sequence is shifted to values $-a$ and a with period M

The autocorrelation function is also periodic and given by

$$R(\tau) = \begin{cases} a^2, & \tau = 0, \pm M, \pm 2M, \ldots \\ -\frac{a^2}{M}, & \text{otherwise} \end{cases}$$

Since $R(\tau)$ is periodic with period M, it has a Fourier representation:

$$R(\tau) = \sum_{k=0}^{M-1} C_k e^{\mathrm{i}2\pi\tau k/M}, \quad \text{with Fourier coefficients } C_k$$

Therefore, the spectrum of PRBS is an impulse train:

$$S(\omega) = \sum_{k=0}^{M-1} C_k \delta(\omega - \frac{2\pi k}{M})$$

Spectral density of PRBS

Hence, the Fourier coefficients

$$C_k = \frac{1}{M} \sum_{\tau=0}^{M-1} R(\tau) e^{-\mathrm{i}2\pi\tau k/M}$$

are also the spectral coefficients of $S(\omega)$

Using the expression of $R(\tau)$, we have

$$C_0 = \frac{a^2}{M^2}, \quad C_k = \frac{a^2}{M^2}(M+1), \quad k = 1, 2, \ldots$$

Therefore,

$$S(\omega) = \frac{a^2}{M^2}\left[\delta(\omega) + (M+1)\sum_{k=1}^{M-1}\delta(\omega - 2\pi k/M)\right]$$

It does not resemble spectral characteristic of a white noise (flat spectrum)

Comparison of the covariances between filtered inputs

Define $y_1(t)$ as the output of a filter:

$$y_1(t) - ay_1(t-1) = u_1(t),$$

with white noise $u(t)$ of zero mean and variance λ^2

Let $y_2(t)$ be the output of the same filter:

$$y_2(t) - ay_2(t-1) = u_2(t),$$

where $u_2(t)$ is a PRBS of period M and amplitude λ

What can we say about the covariances of $y_1(t)$ and $y_2(t)$?

Comparison of the correlations between filtered inputs

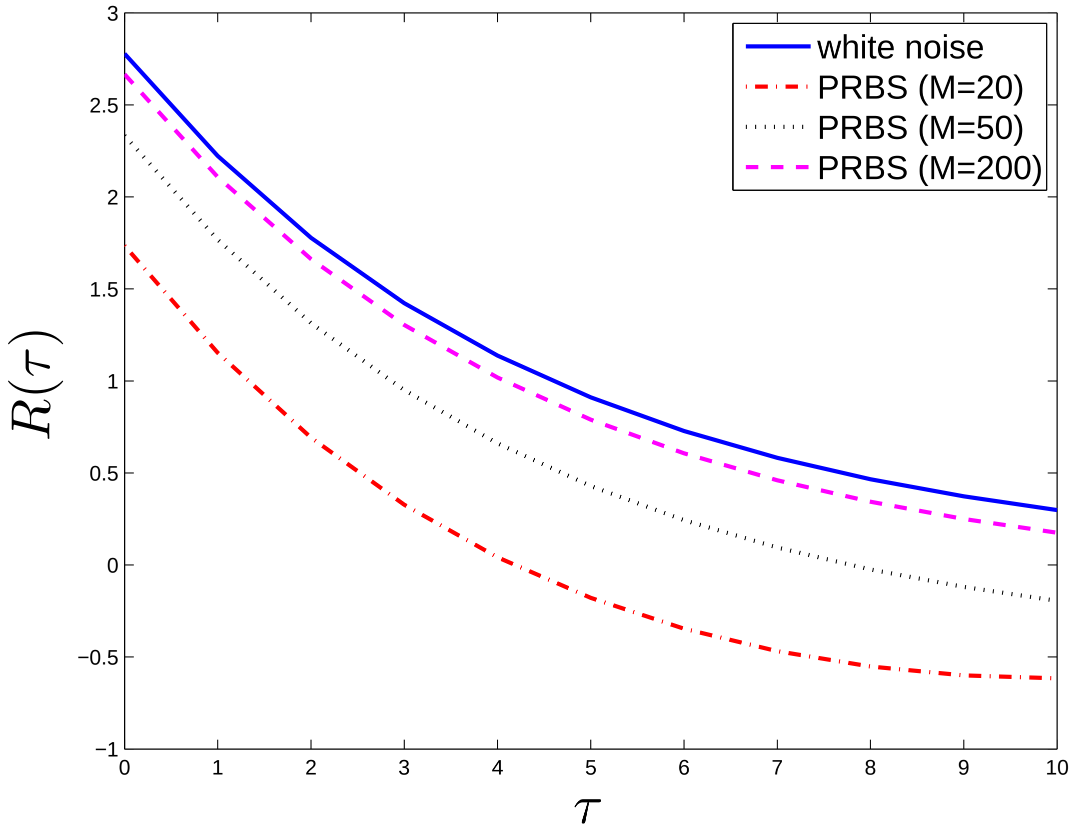
The correlation function of $y_1(t)$ is given by

$$R_1(\tau) = \frac{\lambda^2}{1-a^2} a^\tau, \quad \tau \geq 0$$

The correlation function of $y_2(t)$ can be calculated as

$$\begin{aligned} R_2(\tau) &= \int_{-\pi}^{\pi} S_{y_2}(\omega) e^{\mathrm{i}\omega\tau} d\omega \\ &= \int_{-\pi}^{\pi} S_{u_2}(\omega) \left| \frac{1}{1 - ae^{\mathrm{i}\omega}} \right|^2 e^{\mathrm{i}\tau\omega} d\omega \\ &= \frac{\lambda^2}{M} \left[\frac{1}{(1-a)^2} + (M+1) \sum_{k=1}^{M-1} \frac{\cos(2\pi\tau k/M)}{1 + a^2 - 2a\cos(2\pi k/M)} \right] \end{aligned}$$

Plots of the correlation functions



- The filter parameter is $a = 0.8$
- $R(\tau)$ of white noise and PRBS inputs are very close when M is large

Persistent excitation

A signal $u(t)$ is *persistently exciting* of order n if

1. The following limit exists:

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t+\tau)u(t)^*$$

2. The following matrix is positive definite

$$\mathbf{R}(n) = \begin{bmatrix} R(0) & R(1) & \dots & R(n-1) \\ R(-1) & R(0) & \dots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(1-n) & R(2-n) & \dots & R(0) \end{bmatrix}$$

(if $u(t)$ is from an ergodic stochastic process, then $\mathbf{R}(n)$ is the usual covariance matrix (assume zero mean))

Examining the order of persistent excitation

- **White noise input** of zero mean and variance λ^2

$$R(\tau) = \lambda^2 \delta(\tau), \quad \mathbf{R}(n) = \lambda^2 I_n$$

Thus, white noise is persistently exciting of *all* orders

- **Step input** of magnitude λ

$$R(\tau) = \lambda^2, \forall \tau, \quad \mathbf{R}(n) = \lambda^2 \mathbf{1}_n$$

A step function is persistently exciting of order 1

- **Impulse input**: $u(t) = 1$ for $t = 0$ and 0 otherwise

$$R(\tau) = 0, \forall \tau, \quad \mathbf{R}(n) = 0$$

An impulse is *not* persistently exciting of any order

Example 1: FIR models

Recall the problem of estimating an FIR model where

$$h(k) = 0, \quad k \geq M$$

The coefficients $h(k)$ are the solution to the following equation

$$\begin{bmatrix} R^*_{yu}(0) \\ R^*_{yu}(1) \\ \vdots \\ R^*_{yu}(M-1) \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(1) & \cdots & R_u(M-1) \\ R_u(-1) & R_u(0) & \cdots & R_u(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_u(1-M) & R_u(2-M) & \cdots & R_u(0) \end{bmatrix} \begin{bmatrix} h^*(0) \\ h^*(1) \\ \vdots \\ h^*(M-1) \end{bmatrix}$$

To solve the equation, the matrix $\mathbf{R}(M+1)$ must be nonsingular

Example 2: Estimating noisy linear models

Consider a least-squares problem of estimating a first-order model

$$y(t) = ay(t-1) + bu(t) + e(t)$$

where $u(t)$ is an input signal, and $e(t)$ is an i.i.d. noise of zero mean

We can show that

- If $u(t)$ is a PRBS or step input, the consistent estimates are obtained, i.e.,
$$(\hat{a}, \hat{b}) \to (a, b), \quad \text{as } N \to \infty$$
- If $u(t)$ is an impulse, $\hat{a} \to a$ but $\hat{b}$ *does not* converge to b as N increases
- In loose terms, the impulse input does not provide enough information on $y(t)$ to estimate b

Properties of persistently exciting signals

Let $u(t)$ be a multivariable ergodic process. Assume that $S_u(\omega)$ is positive definite in at least n distinct frequencies (within the interval $(-\pi, \pi)$)

We have the following two properties

Property 1 $u(t)$ is persistently exciting of order n

Property 2 The filtered signal $y(t) = H(q^{-1})u(t)$ is persistently exciting of order n

where $H(z)$ is an asymptotically stable linear filter and $\det H(z)$ has no zero on the unit circle

From above facts, we can imply

An ARMA process is persistently exciting of any finite order

Examining the order of PRBS

Consider a PRBS of period M and magnitude $a, -a$

The matrix containing n-covariance sequenes (where $n \leq M$) is

$$\mathbf{R}(n) = \begin{bmatrix} a^2 & -a^2/M & \dots & -a^2/M \\ -a^2/M & a^2 & \dots & -a^2/M \\ \vdots & \vdots & \ddots & \vdots \\ -a^2/M & -a^2/M & \dots & a^2 \end{bmatrix}$$

Therefore, for any $x \in \mathbf{R}^n$,

$$\begin{aligned} x^T\mathbf{R}(n)x &= x^T\left((a^2 + \frac{a^2}{M})I - \frac{a^2}{M}\mathbf{1}\mathbf{1}^T\right)x \\ &\geq a^2(1+\frac{1}{M})x^Tx - \frac{a^2}{M}x^Tx\mathbf{1}^T\mathbf{1} = a^2\|x\|^2\left(1 + \frac{(1-n)}{M}\right) \geq 0 \end{aligned}$$

A PRBS with period M is persistently exciting of order M

Examining the order of sum of sinusoids

Consider the signal $u(t) = \sum_{k=1}^{m} a_k \sin(\omega_k t + \phi_k)$

where $0 \leq \omega_1 < \omega_2 < \ldots < \omega_m \leq \pi$

The spectral density of u is given by

$$S(\omega) = \sum_{k=1}^{m} \frac{C_k}{2} [\delta(\omega - \omega_k) + \delta(\omega + \omega_k)]$$

Therefore $S(\omega)$ is nonzero (in the interval $(-\pi, \pi]$) in exactly n points where

$$n = \begin{cases} 2m, & 0 < \omega_1, \omega_m < \pi \\ 2m-1, & 0 = \omega_1, \text{ or } \omega_m = \pi \\ 2m-2, & 0 = \omega_1 \text{ and } \omega_m = \pi \end{cases}$$

It follows from Property 1 that $u(t)$ is persistently exciting of order n

Summary

- The choice of input is imposed by the type of identification method
- The input signal should be persistently exciting of a certain order to ensure that the system can be identified
- Some often used signals include PRBS and ARMA processes

References

Chapter 5 in
T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989