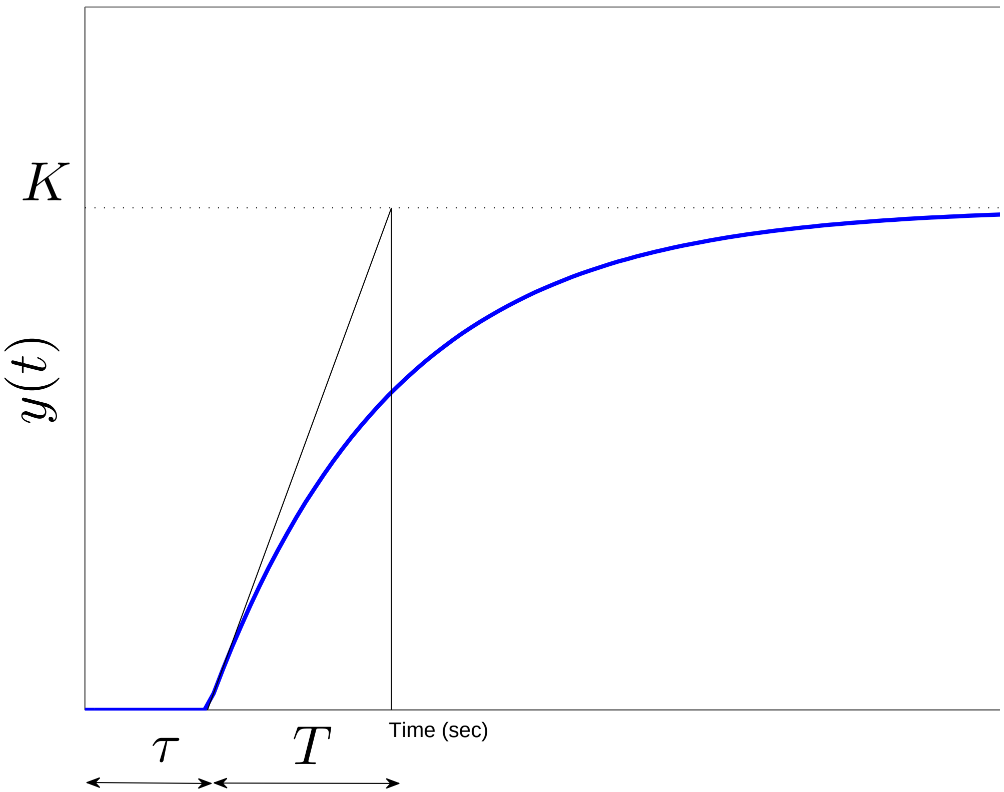
2. Transient and Frequency Analysis

- Step response
- Impulse response
- Basic frequency analysis
- Improved frequency analysis

Step response of a first-order system

$$G(s) = \frac{K}{1+sT}e^{-s\tau}, \quad y(t) = K\left[1 - e^{-(t-\tau)/T}\right]u(t-\tau)$$

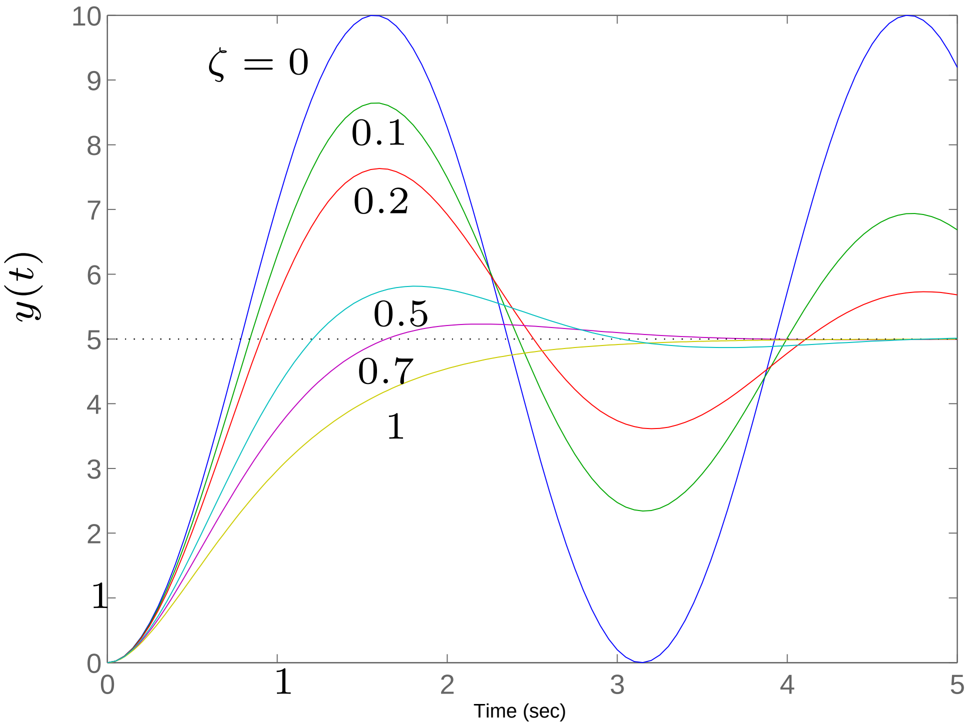
determine K from the final value, T and τ from the steepest tangent



Step response of a second-order system

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

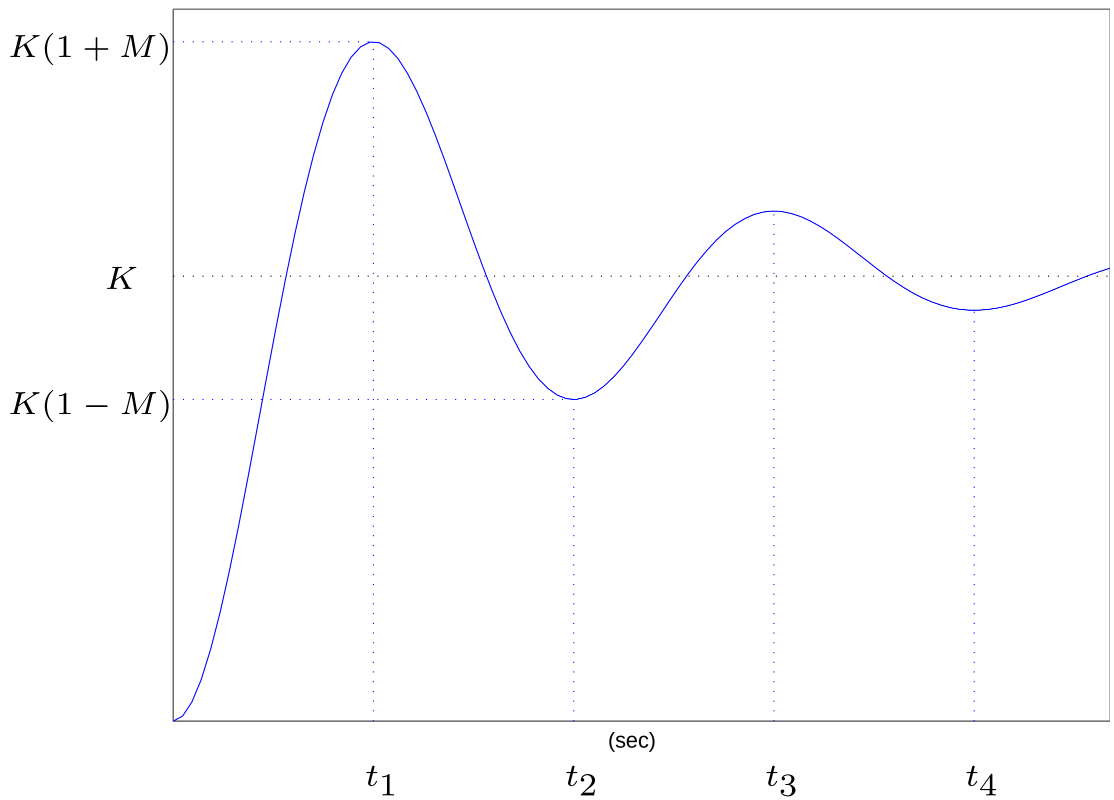
$$y(t) = K\left[1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin(\omega_0\sqrt{1-\zeta^2}t + \tau)\right], \quad \tau = \arccos\zeta$$



Step response of a second-order system (cont.)

The local extrema of the step response occur at

$$t_k = \frac{k\pi}{\omega_0\sqrt{1-\zeta^2}}, \quad k = 1, 2, \ldots \quad \text{and} \quad y(t_k) = K(1 - (-1)^k M^k)$$



Step response of a second-order system (cont.)

Determine ζ from

$$M = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Compute ω_0 when the period T of oscillations are determined

$$\omega_0 = \frac{2\pi}{T\sqrt{1-\zeta^2}}$$

Impulse response

Consider a system described by

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k) + v(t)$$

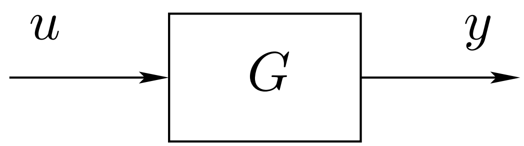
where $v(t)$ is a disturbance and $u(t)$ is an impulse input:

$$u(t) = \begin{cases} \alpha, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

If the noise level is low, the estimate of $g(t)$ is

$$\hat{g}(t) = \frac{y(t)}{\alpha}$$

Basic frequency analysis

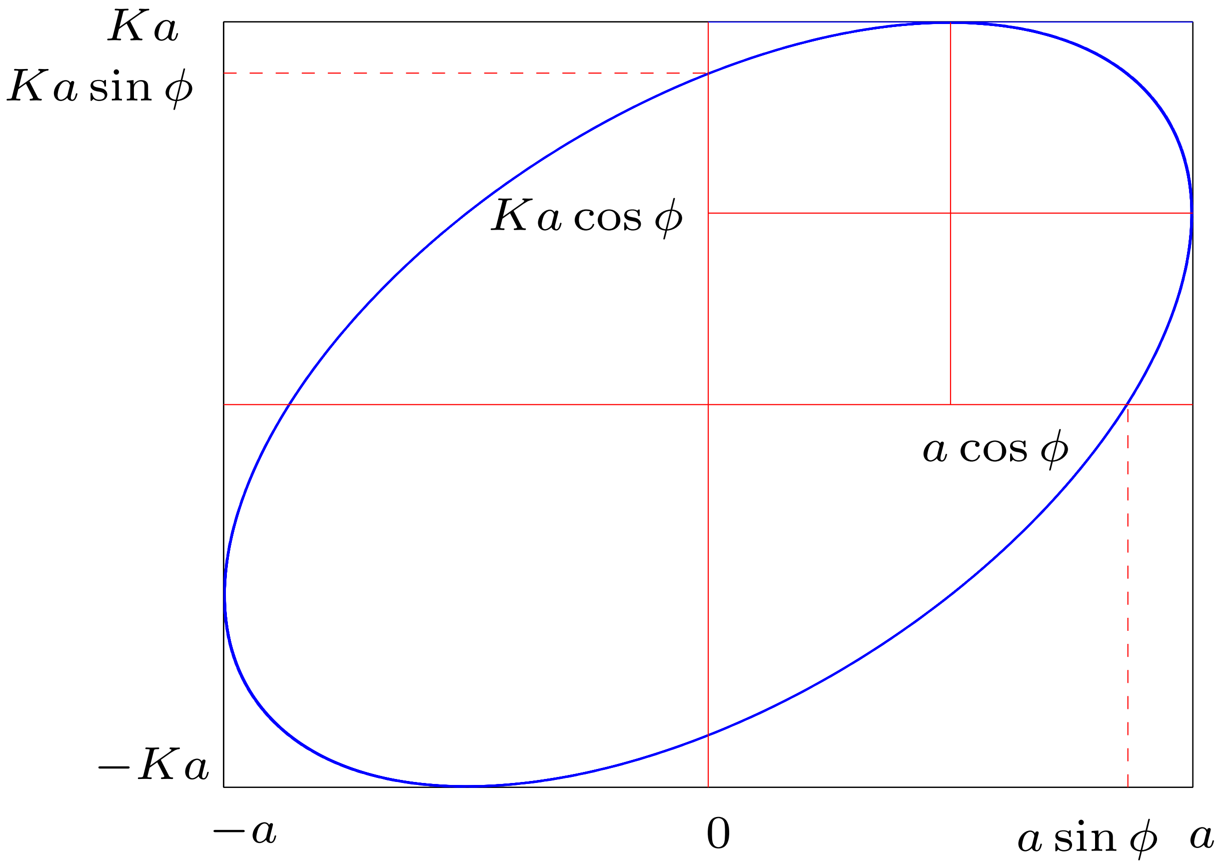


Sine-wave input: $u(t) = a\sin(\omega t)$

$$y(t) = a|G(\mathrm{i}\omega)|\sin(\omega t + \phi) + \text{ transient}$$

- Determine the amplitudes and the phase shift of $y(t)$
- Repeat for a number of ω and obtain a graphical approximation of $G(\mathrm{i}\omega)$
- In the presence of noise, it is difficult to estimate a and ϕ

Determining amplitude and phase



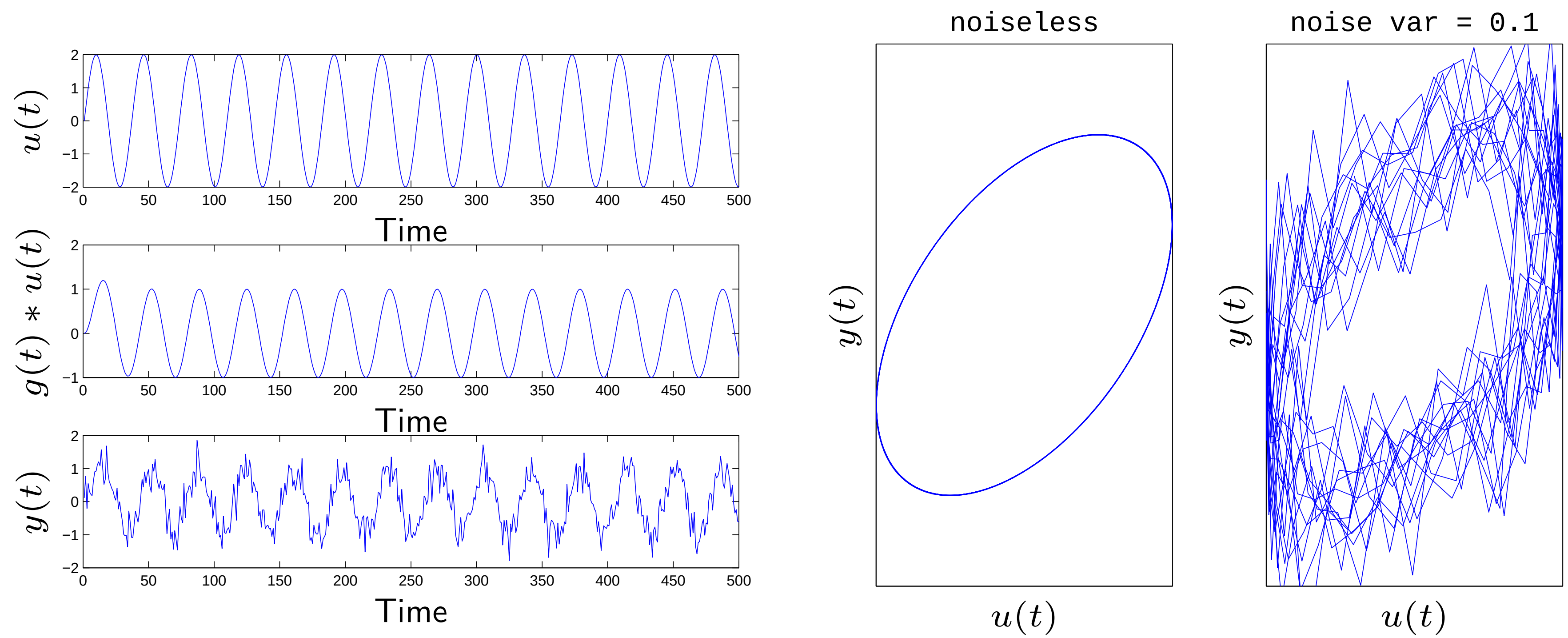
Plot $u(t) = a\sin\omega t$

VS

$y(t) = Ka\sin(\omega t + \phi)$

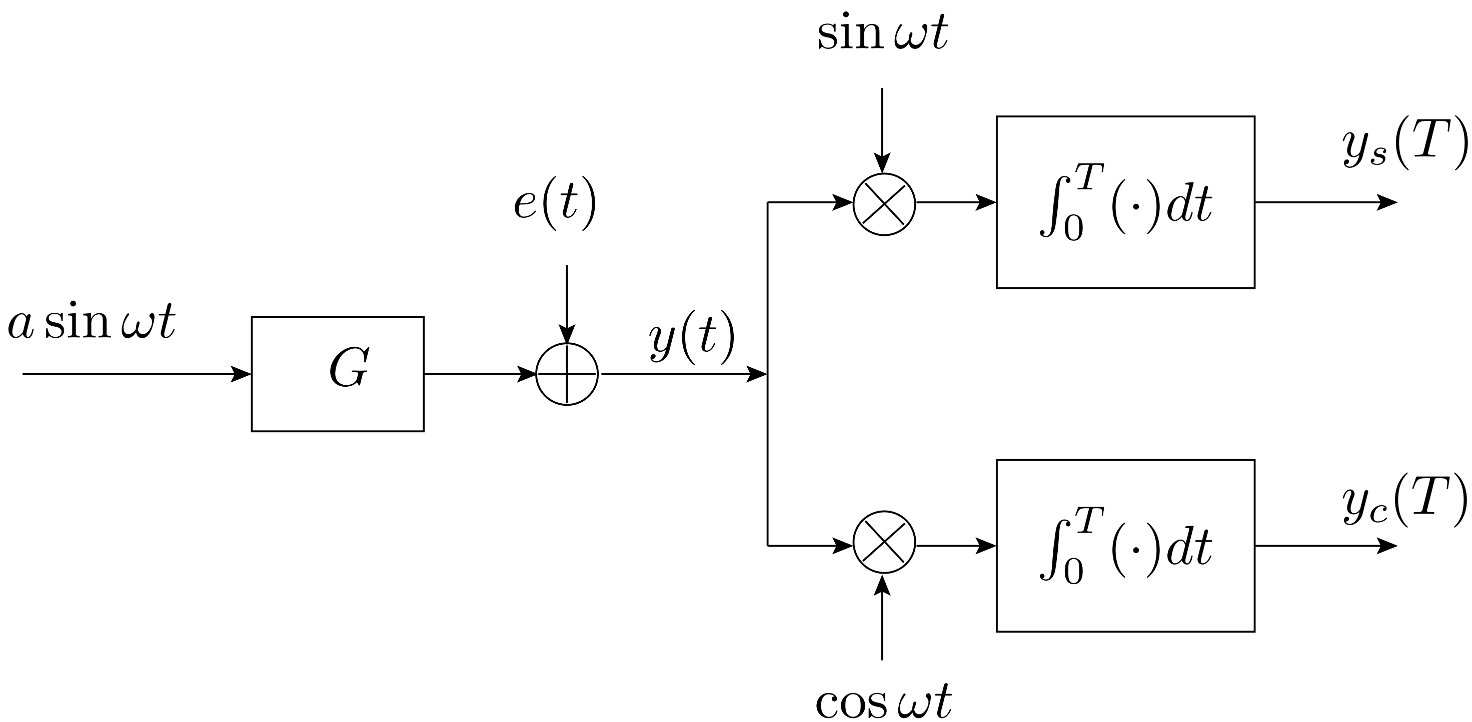
$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} K^2 & -K\cos\phi \\ -K\cos\phi & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = K^2 a \sin^2\phi$$

Effect of noise



- Example with $a = 2, G(s) = 1/(s+1)$ at frequency $\omega_0 = \sqrt{3}$
- Noisy data make it difficult to determine amplitude and phase

Improved frequency analysis



- Suppress the effect of the noise $e(t)$ by taking correlation with a cosine function

Improved frequency analysis (cont.)

If $T = 2k\pi/\omega$, then

$$y_s(T) = \frac{a|G(\mathrm{i}\omega)|T}{2}\cos\phi + \int_0^T e(t)\sin\omega t dt,$$

$$y_c(T) = \frac{a|G(\mathrm{i}\omega)|T}{2}\sin\phi + \int_0^T e(t)\cos\omega t dt$$

- The integral terms can be considered as projections of $e(t)$ on an orthonormal basis
- The estimate of $G(\mathrm{i}\omega)$ is

$$\mathrm{Re}\{\hat{G}(\mathrm{i}\omega)\} = \frac{2y_s(T)}{aT}, \quad \mathrm{Im}\{\hat{G}(\mathrm{i}\omega)\} = \frac{2y_c(T)}{aT}$$

References

Chapter 6 in
L. Ljung, *System Identification: Theory for the User*, Prentice Hall, Second edition, 1999

Chapter 3 in
T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989