2301107 Calculus I
2. Derivatives (II)

Outline

2.4. Derivatives of Trigonometric functions

2.5. The Chain Rule

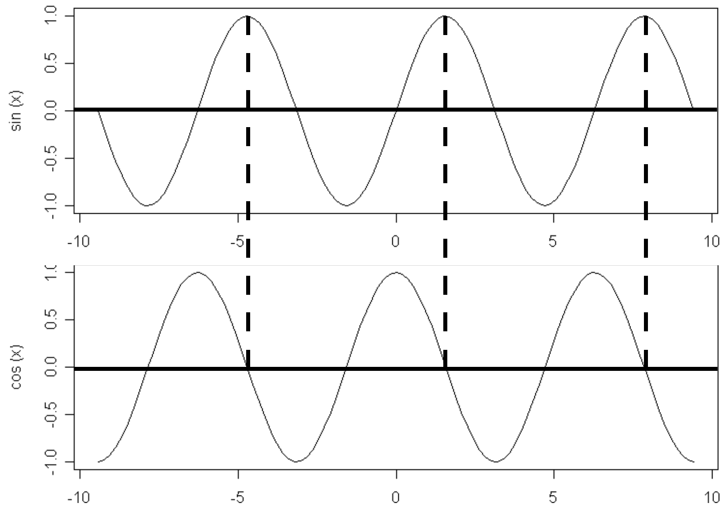
2.6. Implicit Differentiation

2.7. Higher Derivatives

2.8. Related rates

2.9. Linear approximations and differentials

2.4. Derivatives of Trigonometric functions

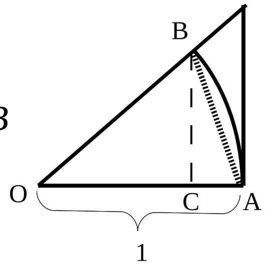


Derivative of sin

- Guess sin'(x) = cos(x)

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]\end{aligned}$$

Limit of $\dfrac{\sin\theta}{\theta}$



- Assume that θ lies between 0 and $\pi/2$.

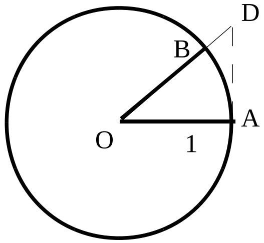
$$|BC| < |AB| < arc\ AB$$

Hence,

$$\sin\theta < \theta.$$

- arc AB < | AD | = | OA | tan θ

$$\theta < \sin\theta / \cos\theta$$



- Taking limit as θ go to 0. We have

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$$

Limit of $\dfrac{\cos\theta - 1}{\theta}$

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\frac{\cos\theta+1}{\cos\theta+1}\right) = \lim_{\theta\to 0}\left(\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}\right)\\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{-\sin\theta}{\cos\theta+1}\\
&= -1\times\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1} = -1\times 0 = 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]\\
&= 0 + \cos x
\end{aligned}$$

Derivative of cos

- Guess cos'(x) = -sin(x)

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h}\\
&= \lim_{h\to 0}\left[\cos x\left(\frac{\cos h-1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\right]\\
&= 0 - \sin x
\end{aligned}$$

Student note

1. Differentiate $y = x^2 \sin(x)$.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos(x)\dfrac{d\sin(x)}{dx} - \sin(x)\dfrac{d\cos x}{dx}}{\cos^2 x}$$

$$= \frac{\cos x\cos x + \sin x\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = sec^2 x$$

$$\frac{d\sin(x)}{dx} = \cos x \qquad \frac{d\,csc(x)}{dx} = -csc\,x\cot x$$

$$\frac{d\cos(x)}{dx} = -\sin x \qquad \frac{d\,sec(x)}{dx} = sec\,x\tan x$$

$$\frac{d\tan(x)}{dx} = sec^2 x \qquad \frac{d\cot(x)}{dx} = -csc^2 x$$

Student note

2. Differentiate $f(x) = \dfrac{sec\,x}{1+\tan x}$.

Student note

3. Calculate the following $\lim_{x\to 0}\dfrac{\sin 7x}{4x}$, $\lim_{x\to 0} x\cot x$.

2.5. The Chain Rule

- If f and g are both differentiable and $F = f\circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

- In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or

$$(f\circ g)' = f'(g(x))g'(x)$$

Student note

4. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Student note

5. Differentiate $y_1 = \sin(x^2)$, $y_2 = \sin^2 x$.

The power rule combined with the chain rule

- If n is any real number and $u = g(x)$ is differentiable, then

Alternatively, $\frac{d}{dx}\left(u^n\right) = n\,u^{n-1}\frac{du}{dx}$

$$\frac{d}{dx}\left(g(x)\right)^n = n\left(g(x)\right)^{n-1} g'(x).$$

Student note

6. Differentiate $y_1 = (x^3 - 1)^{80}$, $y_2 = (2x+1)^5(x^3 - x + 1)^4$,

$$y_3 = \left(\frac{x-2}{2x+1}\right)^9, \quad y_4 = \frac{1}{\sqrt[3]{x^2+x+1}}.$$

Student note

7. Differentiate $f(x) = \sin(\cos(\tan(x)))$, $g(x) = \sqrt{sec\, x^3}$.

Proof of the Chain rule

- Suppose $u = g(x)$ is differentiable at a, and $y = f(u)$ is differentiable at $b = g(a)$.
- Let Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = (g'(a) + \varepsilon_1)\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u = (f'(b) + \varepsilon_2)\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$.

Proof of the Chain rule

By substitution,

$$\Delta y = (f'(b) + \varepsilon_2)(g'(a) + \varepsilon_1)\Delta x$$

Therefore,

$$\frac{\Delta y}{\Delta x} = (f'(b) + \epsilon_2)(g'(a) + \epsilon_1).$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (f'(b) + \epsilon_2)(g'(a) + \epsilon_1)$$
$$= f'(b)\, g'(a) = f'(g(a))\, g'(a).$$

Student note

8. Find derivative of the following functions.

$F(x) = (x^3 + 4x)^7$ $\quad f(x) = (1 + x^4)^{\frac{2}{3}}$ $\quad g(t) = \dfrac{1}{(t^4+1)^3}$

$f(t) = \sqrt[3]{1 + \tan t}$ $\quad y = (2x-5)^4 (8x^2 - 5)^{-3}$ $\quad y = \cot\left(\dfrac{x}{2}\right)$

$y = x^3 \cos nx$ $\quad y = \sin(x \cos x)$ $\quad y = sec^2 x + \tan^2 x$

$y = \sqrt{x + \sqrt{x}}$ $\quad y = \sin(\tan \sqrt{\sin x})$ $\quad y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

Student note

9. Suppose $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$. Find $F'(3)$.

Student note

10. Proof that derivative of an even function is an odd function and vice versa.

($f(-x) = f(x)$ is a even function, $f(-x) = -f(x)$ is an odd function)

2.6. Implicit Differentiation

- Some functions are defined implicitly by a relation between x and y such as (upper or lower semicircle)

$$x^2 + y^2 = 25$$

or (folium of Descartes)

$$x^3 + y^3 = 6xy$$

- There is no need to write the explicit formulation to determine the derivative of function y.

Implicit Differetiation (cont.)

- We can use the method of **implicit differentiation**.
  - Differentiate both sides of the equations with respect to x.
  - Solving the resulting equation for y'.

11. If $x^2 + y^2 = 25$, find y' and then determine an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Student note

12. Find y' if $x^3 + y^3 = 6xy$ and determine the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

At what points on the curve is the tangent line horizontal.

Student note

13. Find y' if $\sin(x + y) = y^2(\cos x)$.

Orthogonal Trajectories

- Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular.
- Two families of curves are **orthogonal trajectories** of each other if every curve in one family is orthogonal to every curve in the other family.

Student note

14. The equation

$$xy = c,\ c \neq 0$$

represents a family of hyperbolas.

The equation

$$x^2 - y^2 = k,\ k \neq 0$$

represent another family of hyperbolas with asymptotes $y = \pm x$. Show that the families are orthogonal trajectories of each other.

Student note

15. Find y'

$$x^2 - y^2 = 1 \qquad 1 + x = \sin(x y^2) \qquad \sqrt{x+y} = 1 + x^2 y^2$$
$$\sqrt{x y} = 1 + x^2 y \qquad x y = \cot(x y) \qquad \tan(x - y) = \frac{y}{1 + x^2}.$$

Student note

16. Show that given curves are orthogonal

$$2x^2 + y^2 = 3, \qquad x = y^2$$
$$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72.$$

Student note

17. Show that families of curves are orthogonal trajectories of each other

$$x^2 + y^2 = r^2, \quad a x + b y = 0$$
$$y = a x^3, \qquad x^2 + 3y^2 = b.$$

2.7. Higher Derivatives

- If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by

$$(f')' = f''$$

The new function is called the **second derivative** of f.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}.$$

Also denoted by $f''(x) = D^2 f(x)$.

Student note

18. If $f(x) = x \cos x$, find and interpret $f''(x)$.

Acceleration

- In general, we can interpret a second derivative as a rate of change of a rate of change, which is called an ***acceleration***.
- If $s = s(t)$ is the position function of an object that moves in a straight line, its first derivative represents the velocity

$$v(t) = s'(t) = \frac{ds}{dt}.$$

This instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object.

$a(t) = v'(t) = s''(t)$.

Student note

19. The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters

(1) Find the acceleration at time t. What is the acceleration after 4 s.?

(2) When is the particle speeding up? When is it slowing down?

The n^{th} derivative

- The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$.
- It is the slope of the curve $y = f''(x)$.

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = \frac{d^3 y}{dx^3} = D^3 f(x).$$

We can defined n^{th} derivative of f, $f^{(n)}$ as

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n f(x).$$

Student note

20. If $y = x^3 - 6x^2 - 5x + 3$, determine $f^{(n)}(x)$?

Student note

21. If $f(x) = \dfrac{1}{x}$, determine $f^{(n)}(x)$?

Student note

22. Find y'' if $x^4 + y^4 = 16$.

Student note

23. Find the first and second derivatives of the functions

(1) $y = x^n$

(2) $f(t) = t^8 - 7t^6 + 2t^4$

(3) $y = \cos 2x$

(4) $H(t) = \tan 3t$

(5) $h(x) = \sqrt{x^2+1}$

(6) $g(s) = s^2 \cos s$

(7) $h(u) = \dfrac{1-4u}{1+3u}$

(8) $y = \dfrac{4x}{\sqrt{x+1}}$

Student note

24. Determine y'' by implicit differentiation

(1) $9x^2 + y^2 = 9$ (2) $\sqrt{x}+\sqrt{y}=1$

Student note

25. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

Student note

26. Determine the first, second, third derivative of

$$f(x)=\frac{1}{(x^2+x)}$$

Then use the identity,

$$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$$

to compute the derivatives.

2.8. Related rates

- Note that if we are pumping air into a balloon, both the volume and radius are increasing and their rates of increase are related to each other.
- In related rates problem, we want to compute the related rate of one quantity in terms of the rate of change of another quantity (may be easily measured).
- Procedure:
  - Determine the equation that relates two quantities
  - Use the Chain Rule to differentiate both sides
  - Rewrite one related rate in term of others.

Student notes

27. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm.?

Student notes

28. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Student notes

29. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Student notes

30. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection.

Student notes

31. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closet to the searchlight?

Student notes

32. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?

Student notes

33. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

2.9. Linear approximations and differentials

- At the beginning when we try to compute the tangent line at a point a, we use the calculator to approximate the slope of the tangent line.
- For a tangent at $(a, f(a))$, we can approximate $f(x)$ when x is near a as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \approx \frac{f(x) - f(a)}{x - a}$$

$$f(x) = f(a) + f'(a)(x - a).$$

Linear approximations

- Hence,

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Student notes

34. Suppose that after you stuff a turkey its temperature is 50°F and you then put it in a 325°F oven. After an hour the meat thermometer indicates that the temperature of the turkey is 93°F and after two hours it indicates 129°F. Predict the temperature of the turkey after three hours.

Student notes

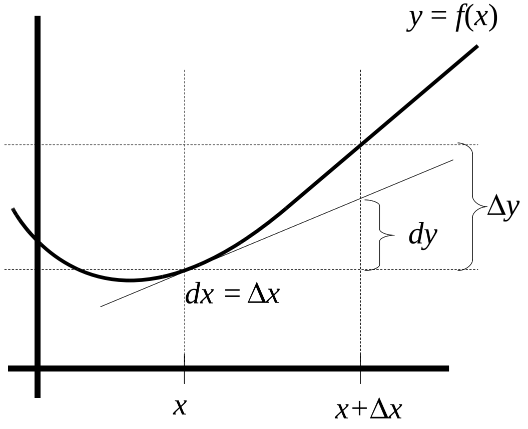
35. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Differentials

- The ideas behind the linear approximations are sometimes formulated in the terminology and notation of *differentials*.
- If $y = f(x)$ is a differentiable function, then the **differential** dx is an independent variable and the **differential** dy is defined in term of dx as

$$dy = f'(x)dx$$

Interpretation of a differential



Student notes

36. Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes

(a) from 2 to 2.05 and

(b) from 2 to 2.01.

Student notes

37. The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

Student notes

38.The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

(a) Use differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error? What is the percentage error?