

2301107 Calculus I

3. Integrations

Chapter 2: Derivatives

C03-2

Outline

- 3.1. Antiderivative and indefinite integral
- 3.2. The definite integral
- 3.3. Calculating areas using Riemann sum
- 3.4. The Fundamental theorem of calculus

3.1. Antiderivatives

- **Definition:** A function F is called an **antiderivative** of f on a interval I if $F'(x) = f(x)$ for all x in I .
- **Theorem:** If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Student notes

1. Find the most general antiderivative of each of the following functions.

$$f_1(x) = \sin x, f_2(x) = x^n, f_3(x) = x^{-3}$$

Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$c f(x)$	$c F(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos(x)$
$x^n(n\neq-1)$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

Student note

2. Find all functions g such that

$$g'(x)=4\sin x+\frac{2x^5-\sqrt{x}}{x}$$

Student note

3. Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

Student note

4. Find f if $f'(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Student note

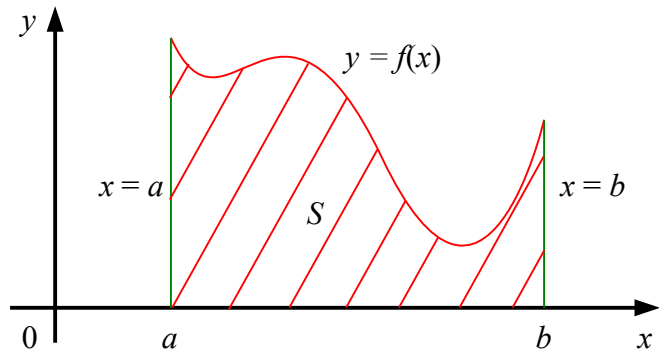
5. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Student note

6. Find the most general antiderivative of the following functions.
- (a) $f(x) = x^{20} + 4x^{10} + 8$ (b) $f(x) = 6\sqrt{x} - \sqrt[9]{x}$
(c) $f(t) = 4\sqrt{t} - \sec t \tan t$ (d) $f(x) = 3 \cos t - 4 \sin t$

Areas

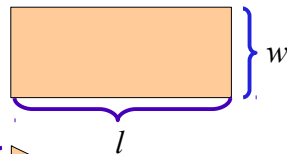
- Determine the area of the region S that lies under the curve $y = f(x)$ from a to b , where $f(x) \geq 0$.



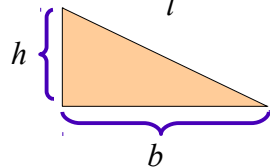
Use the area of known geometries to approximate this!

Area of simple geometries

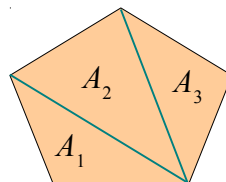
- Area of a rectangle: $A = lw$



- Area of a triangle: $A = \frac{bh}{2}$



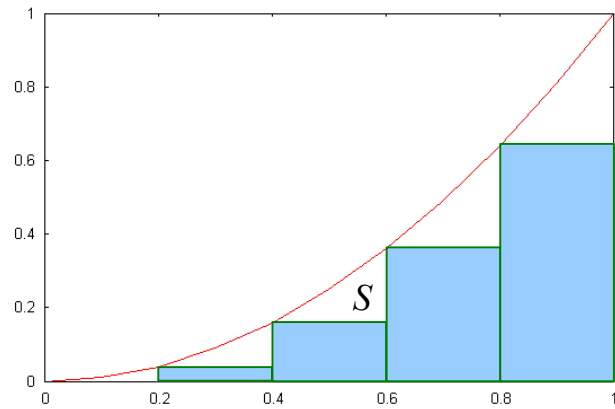
- Area of a polygon: $A = A_1 + A_2 + A_3$



We can compute the exact area of polygons.
Could we extend this to a general shape?

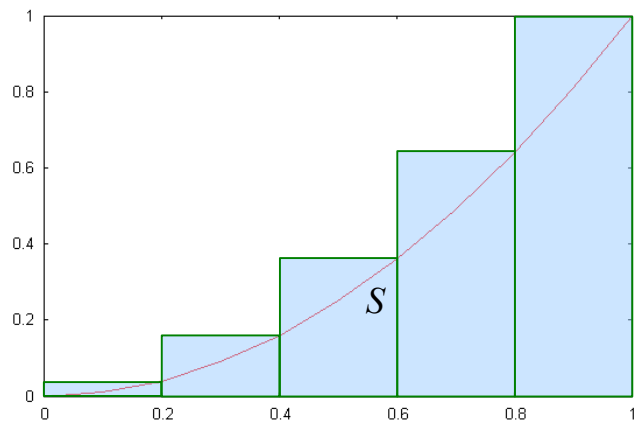
Lower approximation of $y=x^2$ by small rectangles (under approximation)

$$0.2 \cdot 0.2^2 + 0.2 \cdot 0.4^2 + 0.2 \cdot 0.6^2 + 0.2 \cdot 0.8^2 = 0.24$$



Upper approximation of $y=x^2$ by large rectangles (over approximation)

$$0.2 \cdot 0.2^2 + 0.2 \cdot 0.4^2 + 0.2 \cdot 0.6^2 + 0.2 \cdot 0.8^2 + 0.2 \cdot 1^2 = 0.44$$



The effect of increasing n

- For $n = 5$ intervals,
 $0.24 < S < 0.44$
- For $n = 10$ intervals,
 $0.285 < S < 0.385$
- For $n = 100$ intervals,
 $0.32835 < S < 0.33825$
- For $n = 1000$ intervals,
 $0.33283 < S < 0.33383$
- For $n = 10000$ intervals,
 $0.33328 < S < 0.33338$

As n grows larger, both lower & upper approximation converge to an exact area.

The convergence of the lower approximation

- Let L_n be the lower approximation using the left end points

$$\begin{aligned}
 L_n &= 0 + \frac{1}{n} \left(\frac{1}{n} \right)^2 + \frac{1}{n} \left(\frac{2}{n} \right)^2 + \frac{1}{n} \left(\frac{3}{n} \right)^2 + \cdots + \frac{1}{n} \left(\frac{n-1}{n} \right)^2 \\
 &= \frac{1}{n^3} (1^2 + 2^2 + \cdots + (n-1)^2) \\
 &= \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) \\
 \lim_{n \rightarrow \infty} L_n &= \lim_{n \rightarrow \infty} \frac{(n-1)(2n-1)}{6n^2} = \lim_{n \rightarrow \infty} \frac{\left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right)}{6} = \frac{2}{6} = \frac{1}{3}
 \end{aligned}$$

The convergence of the upper approximation

- Let R_n be the upper approximation using the right end points.

$$\begin{aligned}
 R_n &= \frac{1}{n} \left(\frac{1}{n} \right)^2 + \frac{1}{n} \left(\frac{2}{n} \right)^2 + \frac{1}{n} \left(\frac{3}{n} \right)^2 + \cdots + \frac{1}{n} \left(\frac{n}{n} \right)^2 \\
 &= \frac{1}{n^3} (1^2 + 2^2 + \cdots + n^2) \\
 &= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2} \\
 \lim_{n \rightarrow \infty} R_n &= \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right)}{6} = \frac{2}{6} = \frac{1}{3}
 \end{aligned}$$

Chapter 3: Integrations

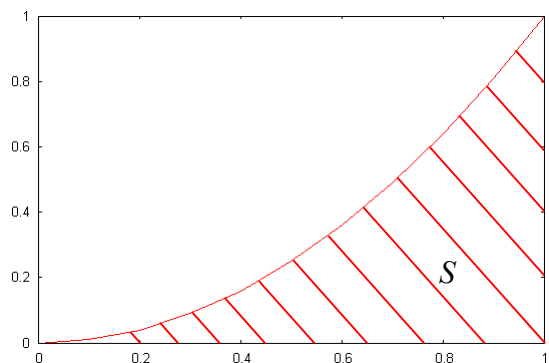
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Area under curve

- Therefore, for any n , $L_n \leq S \leq R_n$, $S = \frac{1}{3}$



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Area definition

- Definition: The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left(f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \right)$$

Distance problem

- The distance problem: Find the distance traveled by an object during a certain period if the velocity of the object is known and fixed in each interval.

$$\text{distance} = \text{velocity} \times \text{time}$$

- Consider time & velocity recording:

Time	0	5	10	15	20	25	30
Velocity	25	31	35	43	47	46	41

- Sol Estimate the total distance traveled as

$$25 \times 5 + 31 \times 5 + 35 \times 5 + 43 \times 5 + 47 \times 5 + 46 \times 5 = 1135$$

Student notes

7. Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$.

Student notes

8. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$.

Student notes

9. Speedometer readings for a motorcycle at 12-second intervals are given in the table (m per min).

Estimate the distance traveled

t:	0	12	24	36	48	60
v:	30	28	25	22	24	27

3.2. The definite integral

- **Definition:** If f is a continuous function defined for $a \leq x \leq b$, divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.
 - let $x_0 (=a), x_1, x_2, \dots, x_n (=b)$ be the endpoints
 - let $x_1^*, x_2^*, x_3^*, \dots, x_n^*$ be any **sample points** in subintervals, $x_i^* \in [x_{i-1}, x_i]$.

Then the **definite integral** of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

This is called a Riemann sum.

Notation

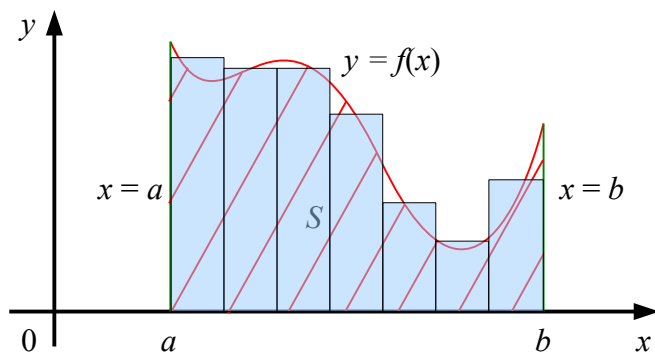
A diagram showing the notation for a Riemann sum with callouts for its components:

- Upper limit**: points to b
- Integrand**: points to $f(x)$
- Integral sign**: points to $\int$
- Lower limit**: points to a
- Riemann sum**: points to $\sum_{i=1}^n f(x_i^*) \Delta x$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

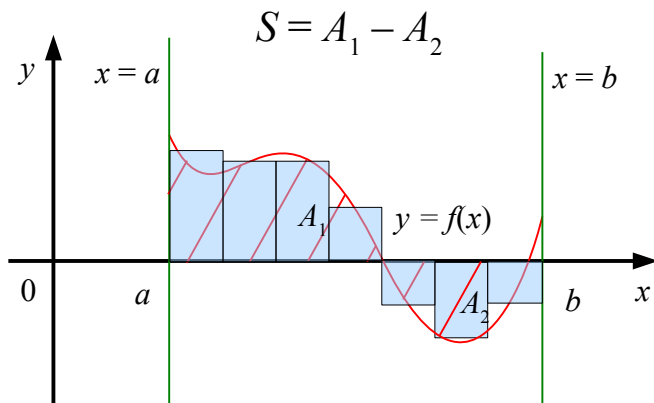
3.3. Riemann sum

- If $f(x) \geq 0$, the Riemann sum is the sum of areas of rectangles under the function f and above the x -axis.



Riemann sum

- If $f(x)$ takes on both positive and negative, the Riemann sum is the net area, that is the difference of areas above and under the x -axis:



Properties of the Definite integral

- For a continuous function f

$$\int_b^a f(x) dx = -\int_a^b f(x) dx, \quad \int_a^a f(x) dx = 0$$

- For a continuous function f and g ,

$$\begin{aligned} \int_a^b c dx &= c(b-a) & \int_a^b f(x) + g(x) dx &= \int_a^b f(x) dx + \int_a^b g(x) dx \\ \int_a^b c f(x) dx &= c \int_a^b f(x) dx & \int_a^b f(x) - g(x) dx &= \int_a^b f(x) dx - \int_a^b g(x) dx \end{aligned}$$

Properties of the Definite integral

- For a continuous function f and $c \in [a, b]$,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

- Comparison properties:

$$\text{If } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x) dx \geq 0$$

$$\text{If } f(x) \geq g(x) \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x) dx \geq \int_a^b g(x) dx$$

$$\text{If } m \leq f(x) \leq M \text{ for } a \leq x \leq b, \text{ then } m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

3.4. The Fundamental Theorem of Calculus PART I

Part I: If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b)

and $g'(x) = f(x)$.

Student notes

10. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2} dt$

Student notes

11. Find $\frac{d}{dx} \int_1^{x^4} \sec t dt$

The Fundamental Theorem of Calculus PART II

Part II: If f is continuous on $[a, b]$, then

$$\int_a^b f(t) dt = F(b) - F(a)$$

where F is any antiderivative of f , that is, a function such that $F' = f$.

Student note

12. Evaluate the integral $\int_{-2}^1 x^3 dx$

Student note

13. Find the area under the parabola $y = x^2$ from 0 to 1.

Student note

14. Find the area under the cosine curve from 0 to b , where $0 \leq b \leq \pi/2$.

Student note

15. Could we determine the integral of $\int_2^{-3} \frac{1}{x^2} dx$?

Differentiation and integration as inverse process

The Fundamental Theorem of Calculus: Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t) dt$ then $g'(x) = f(x)$
2. $\int_a^b f(x) dx = F(b) - F(a)$, where F is any antiderivative of f .

$$\frac{d}{dx} \int_a^x f(t) dt = f(x), \quad \int_a^b f'(x) dx = F(b) - F(a)$$

Student note

16. Find derivatives of the function

$$\int_0^x \sqrt{1+2t} \, dt, \int_2^x t^2 \sin t \, dt, \int_1^{\cos x} (t + \sin t) \, dt$$

Student note

17. Evaluate the integral $\int_{-1}^3 x^5 \, dx, \int_{-2}^5 6 \, dx, \int_2^8 (4x+3) \, dx$

Student note

18. Evaluate the integral $\int_1^8 \sqrt[3]{x} dx, \int_{\pi}^{2\pi} \cos \theta d\theta, \int_{-5}^5 \frac{2}{x^3} dx$

Indefinite integrals

- The indefinite integral is the antiderivative of f

$$\int f(x) dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

- It represents a family of functions that has the same derivatives.

$$\int c f(x) dx = c \int f(x) dx \quad \int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$$

$$\int k dx = kx + C \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$\int \sin x dx = -\cos x + C \quad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \quad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \quad \int \csc x \cot x dx = -\csc x + C$$

Student note

19. Find the general indefinite integral:

$$\int 10x^4 - 2\sec^2 x \, dx$$

Student note

20. Evaluate $\int \frac{\cos \theta}{\sin^2 \theta} d\theta$, $\int_0^{12} (x - 12 \sin x) dx$

Student note

21. Evaluate $\int_0^3 x^3 - 6x \, dx$, $\int_0^1 t^2 - t - 1 \, dt$

Applications

- The Net Change theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

Student note

22. Find the general indefinite integral

$$\int x^{-\frac{3}{4}} dx, \int \sqrt[3]{x} dx, \int x(1+2x^4) dx$$

Student note

23. Find the general indefinite integral

$$\int \frac{\sin x}{1 - \sin^2 x} dx, \int x \sqrt{x} dx, \int (\cos x - 2 \sin x) dx$$

Substitution rule

- The substitution rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

- If $u = g(x)$ then $du = g'(x) dx$. The substitution rule states that it is permissible to operate with dx and du after the integral signs as if they were differentials.

Student note

24. Find $\int x^3 \cos(x^4 + 2) dx$

Student note

25. Evaluate $\int \sqrt{2x+1} \, dx$

Student note

26. Find $\int \frac{x}{\sqrt{1-4x^2}} \, dx$

Student note

27. Calculate $\int \cos 5x \, dx$

Student note

28. Find $\int x^5 \sqrt{1+x^2} \, dx$

Definite integral

- The substitution rule for definite integrals: If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$ then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Student note

29. Evaluate $\int_0^4 \sqrt{2x+1} dx$

Symmetry

- Integrals of Symmetric Functions: Suppose f is continuous on $[-a, a]$.

(a) If f is even [$f(-x) = f(x)$], then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

(b) If f is odd [$f(-x) = -f(x)$], then $\int_{-a}^a f(x) dx = 0$

Student note

30. Evaluate $\int_0^1 \frac{1}{(3-6x)^2} dx$

Definite integral

31. Evaluate the integral of the odd function

$$\int_{-1}^1 \frac{\tan x}{1+x^2+x^4} dx$$

Student note

32. Evaluate $\int \cos 3x \, dx$, $\int x(4-x^2)^{10} \, dx$, $\int x^2 \sqrt{x^3+1} \, dx$

Student note

33. Evaluate $\int \frac{4}{(1+2x)^3} dx$, $\int (2-x)^6 dx$, $\int y^3 \sqrt{2y^4-1} dy$

Student note

34. Evaluate $\int_0^2 (x-1)^{21} dx$, $\int_0^7 \sqrt{4+3x} dx$, $\int_0^\pi \sec^2\left(\frac{t}{4}\right) dt$

Student note

35. Evaluate $\int_0^4 \frac{x}{\sqrt{1+2x}} dx$, $\int_0^4 \frac{1}{(x-2)^3} dx$, $\int_{-a}^a x \sqrt{x^2+a^2} dx$

Student note

36. If f is continuous and $\int_0^4 f(x) dx = 10$, find $\int_0^2 f(2x) dx$