2301107 Calculus I
3. Integrations

Outline

3.1. Antiderivative and indefinite integral

3.2. The definite integral

3.3. Calculating areas using Riemann sum

3.4. The Fundamental theorem of calculus

3.1. Antiderivatives

- Definition: A function F is called an **antiderivative** of f on a interval I if $F'(x) = f(x)$ for all x in I.
- **Theorem**: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Student notes

1. Find the most general antiderivative of each of the following functions.

$$f_1(x) = \sin x, f_2(x) = x^n, f_3(x) = x^{-3}$$

Table of Antidifferentiation Formulas

Function	Particular antiderivative
$c\,f(x)$	$c\,F(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n\ (n\neq -1)$	$\dfrac{x^{n+1}}{n+1}$

Function	Particular antiderivative
$\cos x$	$\sin x$
$\sin x$	$-\cos(x)$
$sec^2 x$	$\tan x$
$sec\,x \tan x$	$sec\,x$

Student note

2. Find all functions g such that

$$g'(x)=4\sin x+\frac{2x^5-\sqrt{x}}{x}$$

Student note

3. Find f if $f'(x)=x\sqrt{x}$ and $f(1) = 2$.

Student note

4. Find f if $f'(x) = 12x^2+6x–4$, $f(0) = 4$, and $f(1) = 1$.

Student note

5. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Student note

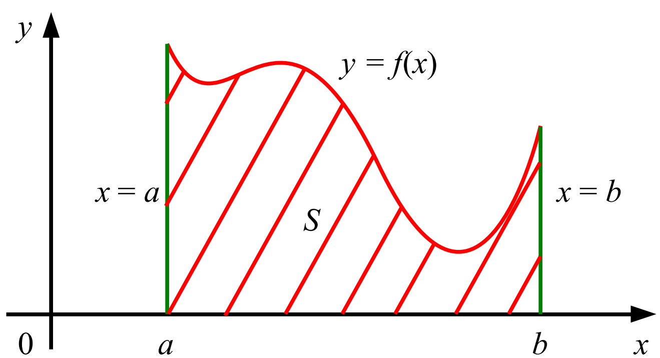
6. Find the most general antiderivative of the following functions.

$(a)\quad f(x) = x^{20} + 4x^{10} + 8 \qquad (b)\quad f(x) = 6\sqrt{x} - \sqrt[6]{x}$

$(c)\quad f(t) = 4\sqrt{t} - \sec t \tan t \qquad (d)\quad f(x) = 3\cos t - 4\sin t$

Areas

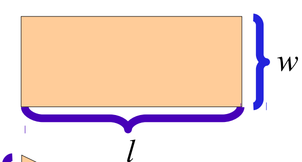
- Determine the area of the region S that lies under the curve $y = f(x)$ from a to b, where $f(x) \geq 0$.



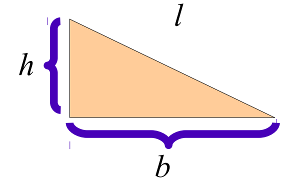
Use the area of known geometries to approximate this!

Area of simple geometries

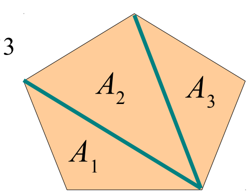
- Area of a rectangle: $A = lw$



- Area of a triangle: $A = \dfrac{bh}{2}$



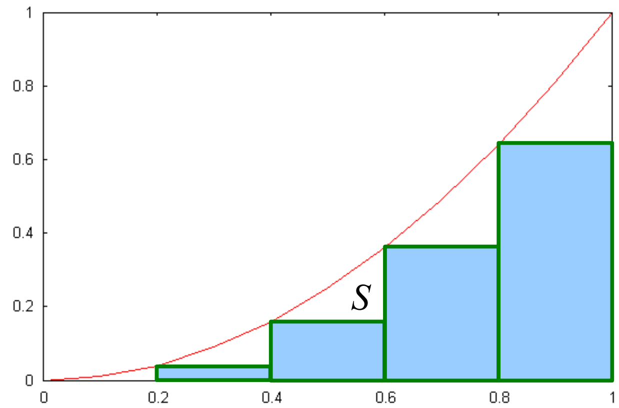
- Area of a polygon: $A = A_1 + A_2 + A_3$



We can compute the exact area of polygons.
Could we extend this to a general shape?

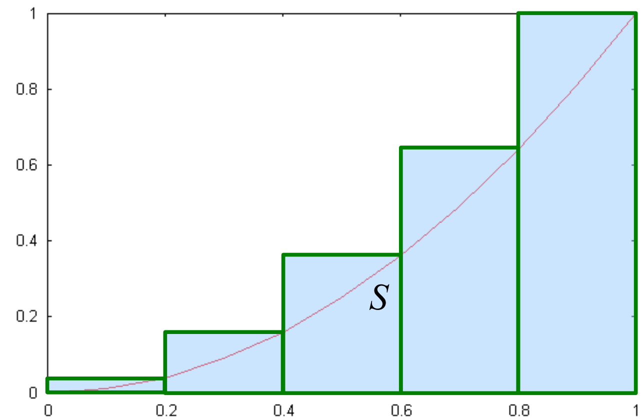
Lower approximation of $y=x^2$ by small rectangles (under approximation)

$0.2*0.2^2 + 0.2*0.4^2 + 0.2*0.6^2+0.2*0.8^2 = 0.24$



Upper approximation of $y=x^2$ by large rectangles (over approximation)

$0.2*0.2^2 + 0.2*0.4^2 + 0.2*0.6^2+0.2*0.8^2 + .2*1^2 = 0.44$



The effect of increasing n

- For n = 5 intervals,
  $0.24 < S < 0.44$
- For n = 10 intervals,
  $0.285 < S < 0.385$
- For n = 100 intervals,
  $0.32835 < S < 0.33825$
- For n = 1000 intervals,
  $0.33283 < S < 0.33383$
- For n = 10000 intervals,
  $0.33328 < S < 0.33338$

As n grows larger, both lower & upper approximation converge to an exact area.

The convergence of the lower approximation

- Let L_n be the lower approximation using the left end points

$$L_n = 0+\frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\frac{1}{n}\left(\frac{3}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n-1}{n}\right)^2$$

$$=\frac{1}{n^3}\left(1^2+2^2+\cdots+(n-1)^2\right)$$

$$=\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}=\frac{(n-1)(2n-1)}{6n^2}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}=\frac{2}{6}=\frac{1}{3}$$

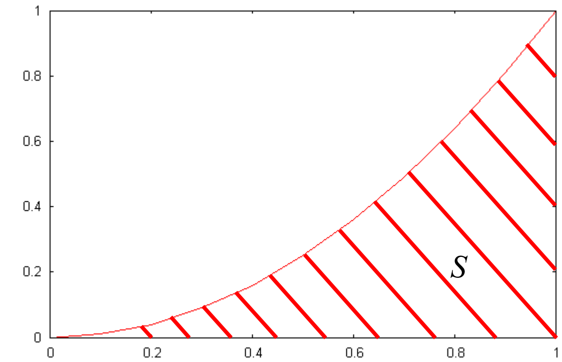
The convergence of the upper approximation

- Let R_n be the upper approximation using the right end points.

$$R_n=\frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\frac{1}{n}\left(\frac{3}{n}\right)^2+\cdots+\frac{1}{n}\left(\frac{n}{n}\right)^2$$
$$=\frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)$$
$$=\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}=\frac{(n+1)(2n+1)}{6n^2}$$
$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}=\frac{2}{6}=\frac{1}{3}$$

Area under curve

- Therefore, for any n, $L_n \le S \le R_n$, $S=\frac{1}{3}$



Area definition

- Definition: The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\left(f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_n)\Delta x\right)$$

Distance problem

- The distance problem: Find the distance traveled by an object during a certain period if the velocity of the object is known and fixed in each interval.

  distance = *velocity* × *time*

- Consider time & velocity recording:

Time	0	5	10	15	20	25	30
Velocity	25	31	35	43	47	46	41

- Sol Estimate the total distance traveled as

$$25\times 5 + 31\times 5 + 35\times 5 + 43\times 5 + 47\times 5 + 46\times 5 = 1135$$

Student notes

7. Estimate the area under the graph of $f(x) = \dfrac{1}{x}$ from $x = 1$ to $x = 5$.

Student notes

8. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$.

Student notes

9. Speedometer readings for a motorcycle at 12-second intervals are given in the table (m per min).

Estimate the distance traveled

t:	0	12	24	36	48	60
v:	30	28	25	22	24	27

3.2. The definite integral

- Definition: If f is a continuous function defined for $a \le x \le b$, divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$.
  - let x_0 $(=a)$, x_1, x_2, ..., x_n $(= b)$ be the endpoints
  - let $x_1^*, x_2^*, x_3^*, ..., x_n^*$ be any **sample points** in subintervals, $x_i^* \in [x_{i-1}, x_i]$.

  Then the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is called a Riemann sum.

Notation

Upper limit

Integrand

Integral sign

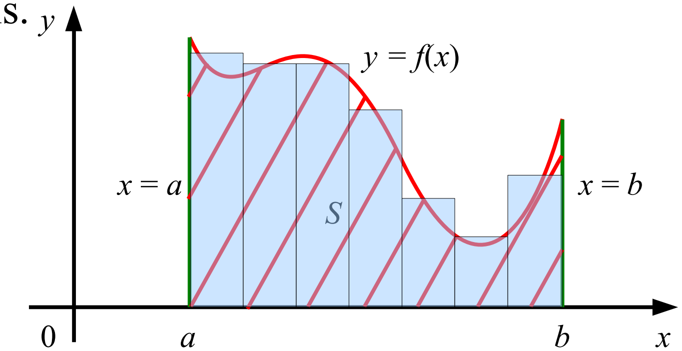
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Lower limit

Riemann sum

3.3. Riemann sum

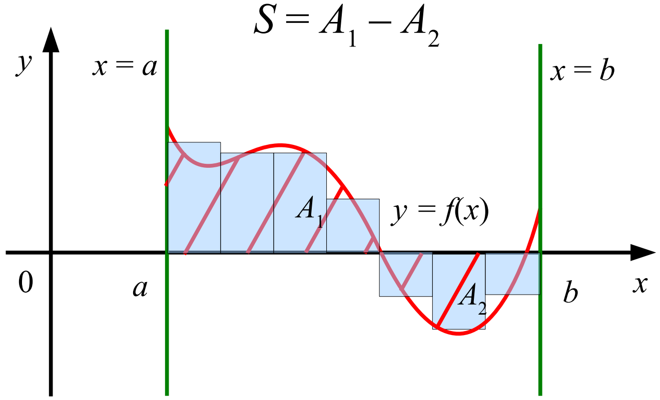
- If $f(x) \geq 0$, the Reimann sum is the sum of areas of rectangles under the function f and above the x-axis.



Riemann sum

- If $f(x)$ takes on both positive and negative, the Reimann sum is the net area, that is the difference of areas above and under the x-axis:

$$S = A_1 - A_2$$



Properties of the Definite integral

- For a continuous function f

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\,, \qquad \int_a^a f(x)\,dx = 0$$

- For a continuous function f and g,

$$\int_a^b c\,dx = c(b-a) \qquad \int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of the Definite integral

- For a continuous function f and $c \in [a, b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- Comparison properties:

$$\text{If} \quad f(x) \geqslant 0 \quad \text{for} \quad a \leqslant x \leqslant b, \text{then} \quad \int_a^b f(x)\,dx \geqslant 0$$

$$\text{If} \quad f(x) \geqslant g(x) \quad \text{for} \quad a \leqslant x \leqslant b, \text{then} \quad \int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

$$\text{If} \quad m \leqslant f(x) \leqslant M \ \text{for}\ a \leqslant x \leqslant b, \text{then} \quad m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

3.4. The Fundamental Theorem of Calculus PART I

Part I: If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \; a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

Student notes

10. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Student notes

11. Find $\frac{d}{dx} \int_1^{x^4} \sec t\,dt$

The Fundamental Theorem of Calculus PART II

Part II: If f is continuous on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Student note

12. Evaluate the integral $\int_{-2}^{1} x^3\,dx$

Student note

13. Find the area under the parabola $y = x^2$ from 0 to 1.

Student note

14. Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$.

Student note

15. Could we determine the integral of $\int_{2}^{-3} \frac{1}{x^2} dx$?

Differentiation and integration as inverse process

The Fundamental Theorem of Calculus: Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_{a}^{x} f(t)\,dt$ then $g'(x) = f(x)$
2. $\int_{a}^{b} f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x),\ \int_{a}^{b} F'(x)\,dx = F(b) - F(a)$$

Student note

16. Find derivatives of the function

$$\int_{0}^{x} \sqrt{1+2t}\,dt,\ \int_{2}^{x} t^2 \sin t\,dt,\ \int_{1}^{\cos x} (t+\sin t)\,dt$$

Student note

17. Evaluate the integral $\int_{-1}^{3} x^5\,dx,\ \int_{-2}^{5} 6\,dx,\ \int_{2}^{8} (4x+3)\,dx$

Student note

18. Evaluate the integral $\int_{1}^{8} \sqrt[3]{x}\, dx$, $\int_{\pi}^{2\pi} \cos\theta\, d\theta$, $\int_{-5}^{5} \frac{2}{x^3}\, dx$

Indefinite integrals

- The indefinite integral is the antiderivative of f

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

- It represents a family of functions that has the same derivatives.

$\int c\, f(x)\,dx = c\int f(x)\,dx$ $\quad$ $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int k\,dx = kx + C$ $\quad$ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$

$\int \sin x\,dx = -\cos x + C$ $\quad$ $\int \cos x\,dx = \sin x + C$

$\int sec^2 x\,dx = \tan x + C$ $\quad$ $\int csc^2 x\,dx = -\cot x + C$

$\int sec\,x \tan x\,dx = sec\,x + C$ $\quad$ $\int csc\,x \cot x\,dx = -csc\,x + C$

Student note

19. Find the general indefinite integral:

$$\int 10x^4 - 2\,sec^2 x\,dx$$

Student note

20. Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$, $\int_{0}^{12} (x - 12\sin x)\,dx$

Student note

21. Evaluate $\int_0^3 x^3 - 6x\,dx$, $\int_0^1 t^2 - t - 1\,dt$

Applications

- The Net Change theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Student note

22. Find the general indefinite integral

$$\int x^{-\frac{3}{4}}\,dx,\ \int \sqrt[3]{x}\,dx,\ \int x(1+2x^4)\,dx$$

Student note

23. Find the general indefinite integral

$$\int \frac{\sin x}{1-\sin^2 x}\,dx,\ \int x\sqrt{x}\,dx,\ \int (\cos x - 2\sin x)\,dx$$

Substitution rule

- The substitution rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

- If $u = g(x)$ then $du = g'(x)\, dx$. The substitution rule states that it is permissible to operate with dx and du after the integral signs as if they were differentials.

Student note

24. Find $\int x^3 \cos(x^4 + 2)\, dx$

Student note

25. Evaluate $\int \sqrt{2x+1}\, dx$

Student note

26. Find $\int \frac{x}{\sqrt{1-4x^2}}\, dx$

Student note

27. Calculate $\int \cos 5x \, dx$

Student note

28. Find $\int x^5 \sqrt{1+x^2}\, dx$

Definite integral

- The substitution rule for definite integrals: If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$ then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Student note

29. Evaluate $\int_0^4 \sqrt{2x+1}\, dx$

Symmetry

- Integrals of Symmetric Functions: Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$

Student note

30. Evaluate $\int_{0}^{1} \frac{1}{(3-6x)^2}\,dx$

Definite integral

31. Evaluate the integral of the odd function

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx$$

Student note

32. Evaluate $\int \cos 3x\,dx$, $\int x(4-x^2)^{10}\,dx$, $\int x^2\sqrt{x^3+1}\,dx$

Student note

33. Evaluate $\int \frac{4}{(1+2x)^3} dx$, $\int (2-x)^6 dx$, $\int y^3 \sqrt{2y^4-1}\, dy$

Student note

34. Evaluate $\int_0^2 (x-1)^{21} dx$, $\int_0^7 \sqrt{4+3x}\, dx$, $\int_0^{\pi} sec^2\left(\frac{t}{4}\right) dt$

Student note

35. Evaluate $\int_0^4 \frac{x}{\sqrt{1+2x}} dx$, $\int_0^4 \frac{1}{(x-2)^3} dx$, $\int_{-a}^{a} x\sqrt{x^2+a^2}\, dx$

Student note

36. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$