

2301107 Calculus I

4. Transcendental functions

Chapter 2: Derivatives

C04--2

Outline

- 4.1. Exponential function
- 4.2. Natural logarithmic function
- 4.3. Trigonometric function
- 4.4. Inverse trigonometric function
- 4.5. Hyperbolic function

4.1. Exponential function

- An exponential function is a function of the form

$$f(x) = a^x$$

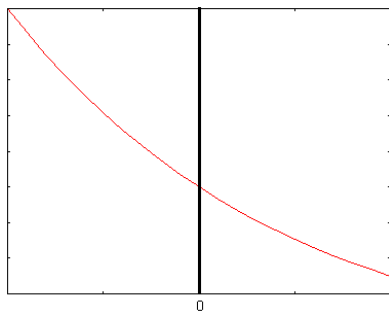
where a is a positive constant.

- If $x = n$, a positive integer, then $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}$
- If $x = 0$, then $a^0 = 1$.
- If $x = -n$, where n is a positive integer, then $a^{-n} = \frac{1}{a^n}$
- If x is a rational number, $x = p/q$ then

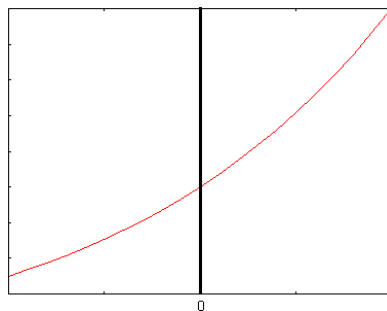
$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

- If x is an irrational number, $a^x = \lim_{r \rightarrow x} a^r$, r rational

Graph of the exponential function



$$y = a^x, 0 < a < 1$$



$$y = a^x, a > 1$$

For $y = 1^x$, we have constant function $y = 1$.

Exponential function

- Theorem: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain = set of real number and range $(0, \infty)$.
- For $a, b > 0$ and any real numbers x, y then

$$a^{x+y} = a^x a^y \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x \quad a^{x-y} = \frac{a^x}{a^y}$$

- Laws of Exponents:

$$\text{If } a > 1 \text{ then } \lim_{x \rightarrow \infty} a^x = \infty, \lim_{x \rightarrow \infty} a^x = 0$$

$$\text{If } 0 < a < 1, \text{ then } \lim_{x \rightarrow \infty} a^x = 0, \lim_{x \rightarrow \infty} a^x = \infty$$

Derivative of exponential function

- For $a > 0$ and $a \neq 1$, the derivative of $f(x) = a^x$ is

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \end{aligned}$$

- We have $f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$
- Therefore, $f(x) = f'(0)a^x$
- The rate of change of any exponential function is proportional to the function itself.

Definition of the Number e

- Definition of the Number e :

e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

- The derivative of the Natural exponential function

$$\frac{d}{dx}(e^x) = e^x.$$

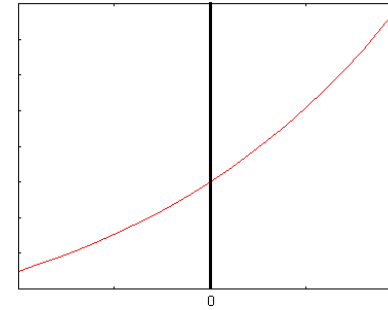
Student note

1. Differentiate the function $y = e^{\tan x}$.

Student note

2. Apply the chain rule where $u = f(x)$, $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$.

Exponential Graph



The exponential function $f(x) = e^x$, is an increasing continuous function on real number with the range $(0, \infty)$. Also,

$$\lim_{x \rightarrow -\infty} e^x = 0, \lim_{x \rightarrow \infty} e^x = \infty.$$

Student note

3. Find the domain of each function

$$f_1(x) = \frac{1}{1+e^x} \quad f_2(x) = \frac{1}{1-e^x} \quad f_3(x) = \sqrt{1-2^x}$$

Student note

4. Differentiate the function

$$f_4(x) = x^2 e^x, f_5(x) = \frac{e^x}{1+x}, f_6(x) = \sqrt{x} e^x.$$

Student note

5. Differentiate the function

$$f_7(u) = e^{\frac{1}{u}}, f_8(x) = e^{k \tan \sqrt{x}}, f_9(x) = \cos(e^{\pi x}).$$

Student note

6. Find y' if $e^{x^2y} = x + y$.

Student note

7. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

Student note

8. Find $f(x)$ if $f''(x) = 3e^x + 5 \sin x$, $f(0) = 1$, $f'(0) = 2$.

4.2. Logarithmic function

- If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one. The inverse is called the logarithmic function with base a , denoted by $\log_a$.

$$f^1(x) = y \leftrightarrow f(y) = x$$

$$\log_a x = y \leftrightarrow a^y = x$$

$$a^{\log_a x} = x$$

- Note that

$$\log_a(a^x) = x$$

Student note

9. Evaluate $\log_3 81$, $\log_{25} 5$ and $\log_{10} 0.0001$.

Logarithmic properties

- Theorem: If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one continuous, increasing function with domain $(0, \infty)$ and range is the set of real numbers.
- If $x, y > 0$ and r is a real number
$$\log_a(xy) = \log_a x + \log_a y \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^r) = r \log_a x$$
- If $a > 1$, then $\lim_{x \rightarrow \infty} \log_a x = \infty$, $\lim_{x \rightarrow 0^+} \log_a x = -\infty$

Student note

10. Evaluate $\log_4 2 + \log_4 32$, $\log_2 80 - \log_2 5$.

Natural logarithm

- The logarithm with base e is called the natural logarithm and has a special notation

$$\log_e x = \ln x$$

- Properties

- $\ln x = y \leftrightarrow e^y = x$
- $\ln(e^x) = x$
- $e^{\ln x} = x, x > 0$
- $\ln e = 1$

- Change of base formula: For any positive number $a \neq 1$,

$$\log_a x = \frac{\ln x}{\ln a}$$

Student note

11. Use the properties of logarithms to expand

$$\log_2 \left(\frac{x^3 y}{z^2} \right), \ln \sqrt{a(b^2 + c^2)}, \ln (uv)^{10}, \ln \frac{3x^2}{(x+1)^5}$$

Student note

12. Solve each equation for x

1) $2 \ln x = 1$

2) $e^{-x} = 5$

3) $e^{3x+1} = k$

4) $7e^x - e^{2x} = 12$

Student note

13. Find the domain and the inverse of the following functions
 $f(x) = \log_2(4x - 2)$, $g(t) = \ln(e^t - 2)$.

Derivative of logarithmic function

- The derivative of the natural logarithm is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

- Properties

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}, \quad \frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$$

Student note

14. Differentiate $y = \ln(x^3 + 1)$, $f(x) = \ln(\sin x)$, $g(x) = \sqrt{\ln x}$.

Student note

15. Find $f'(x)$ if $f(x) = \ln |x|$.

Logarithmic differentiation

- We can simplify the derivatives of complicated functions involving products, quotients or powers using the logarithms, called **logarithmic differentiation**.
- Steps
 - Take logarithms of both sides, $y = f(x)$
 - Differentiate implicitly with respect to x .
 - Solve the resulting equation for y' .
- The power rule: If $f(x) = x^n$ then

$$f'(x) = n x^{n-1}.$$

Student note

16. Differentiate $y = \frac{x^{\frac{3}{4}} \sqrt{x^2 + 1}}{(3x + 2)^5}.$

Logarithmic formula

$$\frac{d}{dx}(a^b) = 0$$

$$\frac{d}{dx}(f(x))^b = b(f(x))^{b-1} f'(x)$$

$$\frac{d}{dx}(a^{g(x)}) = a^{g(x)} \ln a g'(x)$$

- The number e as a limit,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Student note

17. Differentiate $y = x^{\sqrt{x}}$.

Student note

18. Differentiate the following functions

$$f_1(x) = \ln(x^2 + 10) \quad f_2(x) = \frac{1 + \ln x}{1 - \ln x} \quad f_3(x) = \sqrt[5]{\ln x}$$

Student note

19. Differentiate the following functions

$$f_4(x) = \cos(\ln x) \quad f_5(x) = \sqrt{x} \ln x \quad f_6(x) = \ln \sqrt[5]{x}$$

Student note

20. Find y' if $y = \ln(x^2 + y^2)$.

Student note

21. Find y' if $x^y = y^x$.

Integral of transcendental functions

- Consider the integration of an exponential function, because the exponential function $y = e^x$ is its derivative, therefore

$$\int e^x dx = e^x + C$$

Student note

22. Find the area under the curve $y = e^{-3x}$ from 0 to 1.

Student note

23. Evaluate $\int x^2 e^{x^3} dx$

Student note

24. Evaluate the integral $\int_0^5 e^{-3x} dx, \int_0^1 x e^{-x^2} dx, \int e^x \sqrt{1+e^x} dx$

Student note

25. Evaluate the integral $\int \frac{e^x + 1}{e^x} dx$, $\int \frac{e^{\frac{1}{x}}}{x^2} dx$, $\int e^x \sin(e^x) dx$

Integral of logarithmic function

- Note that

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

- Hence,

$$\int \frac{1}{x} dx = \ln|x| + C$$

- Properties

$$\frac{d}{dx} (\log_a x) = \frac{1}{x \ln a} \quad \frac{d}{dx} (a^x) = a^x \ln a \quad \int a^x dx = \frac{a^x}{\ln a} + C$$

Student note

26. Evaluate $\int \frac{x}{x^2+1} dx$, $\int_1^e \frac{\ln x}{x} dx$, $\int \tan x dx$

Student note

27. Evaluate the integral

$$\int_0^5 2^x dx, \int_2^4 \frac{3}{x} dx, \int_1^2 \frac{4+u^2}{u^3} du, \int_1^2 10^t dt, \int x 2^{x^2} dx$$

Table of derivatives of inverse trigonometric function

$$\begin{aligned}\frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} & \frac{d}{dx}(\operatorname{arccsc} x) &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\operatorname{arcsec} x) &= \frac{1}{x\sqrt{x^2-1}} & \frac{d}{dx}(\operatorname{arccot} x) &= -\frac{1}{1+x^2}\end{aligned}$$

Inverse Trigonometric function

- The two most useful integration formulae are

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \arcsin x + C \\ \int \frac{1}{x^2+1} dx &= \arctan x + C\end{aligned}$$

Student note

28. Find $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}} dx$

Student note

29. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}} dx$, $\int \frac{x}{x^4+9} dx$

Student note

30. Evaluate the integral

$$\int_0^1 \frac{4}{t^2+1} dt, \int \frac{1}{\sqrt{1-4t^2}} dt, \int \frac{\arctan x}{1+x^2} dx, \int \frac{t^2}{\sqrt{t-t^6}} dt$$

Hyperbolic function

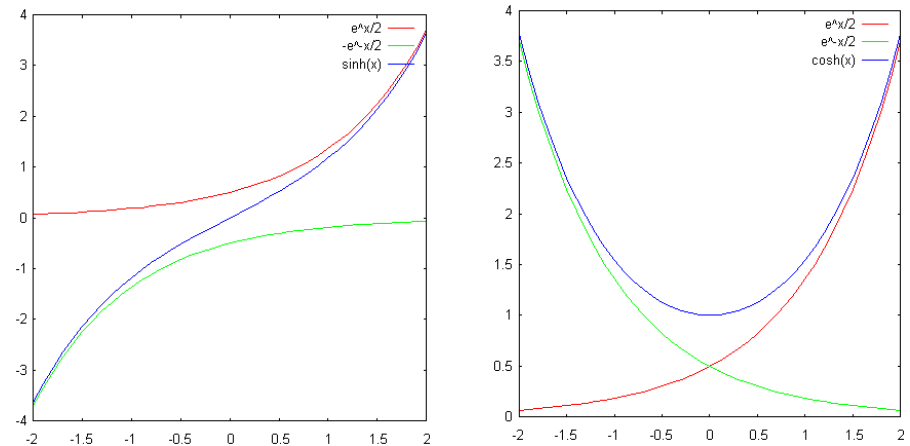
- Certain combination of the exponential functions e^x and e^{-x} arise so frequently in mathematics, which is called hyperbolic function.

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad \operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

Graph of Hyperbolic function



- Graph of $\sinh x$ and $\cosh x$.

Hyperbolic identities

- $\sinh(-x) = -\sinh(x)$ $\cosh(-x) = \cosh(x)$
- $\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

- Derivative of Hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \quad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Integrals of hyperbolic functions

$$\begin{aligned}
 \int \cosh x \, dx &= \sinh x + c & \int \frac{1}{\sqrt{1+x^2}} \, dx &= \operatorname{arcsinh} x + C \\
 \int \sinh x \, dx &= \cosh x + c & \int \frac{1}{\sqrt{x^2-1}} \, dx &= \operatorname{arccosh} x + C \\
 \int \tanh x \, dx &= \operatorname{sech}^2 x + c & \int \frac{1}{1-x^2} \, dx &= \operatorname{arctanh} x + C \\
 \int \operatorname{csch} x \coth x \, dx &= -\operatorname{csch} x + C \\
 \int \operatorname{sech} x \tanh x \, dx &= -\operatorname{sech} x + C \\
 \int \operatorname{csch}^2 x \, dx &= -\coth x + C
 \end{aligned}$$

Student note

31. Evaluate the integral $\int_0^1 \frac{1}{\sqrt{1+x^2}} \, dx \int_0^1 \frac{1}{\sqrt{16t^2+1}} \, dt$

Student note

32. Evaluate the integral

$$\int \sinh x \cosh^2 x \, dx, \int \tanh x \, dx, \int \frac{\cosh x}{\cosh^2 x - 1} \, dx$$

Student note

33. Prove that $\cosh^2 x - \sinh^2 x = 1$ and $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Inverse hyperbolic functions

- The inverse of hyperbolic function

$$\begin{aligned}\sinh^{-1}x &= y \leftrightarrow \sinh y = x, \\ \cosh^{-1}x &= y \leftrightarrow \cosh y = x, \text{ and } y \geq 0 \\ \tanh^{-1}x &= y \leftrightarrow \tanh y = x.\end{aligned}$$

- Formula for compute the inverse hyperbolic

$$\begin{aligned}\sinh^{-1}x &= \ln(x + \sqrt{x^2 + 1}) \\ \cosh^{-1}x &= \ln(x + \sqrt{x^2 - 1}), x \geq 0 \\ \tanh^{-1}x &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).\end{aligned}$$

Derivatives of Inverse hyperbolic functions

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1}x) &= \frac{1}{\sqrt{1+x^2}} & \frac{d}{dx}(\operatorname{csch}^{-1}x) &= -\frac{1}{|x|\sqrt{x^2+1}} \\ \frac{d}{dx}(\cosh^{-1}x) &= \frac{1}{\sqrt{x^2-1}} & \frac{d}{dx}(\operatorname{sech}^{-1}x) &= -\frac{1}{x\sqrt{1-x^2}} \\ \frac{d}{dx}(\tanh^{-1}x) &= \frac{1}{1-x^2} & \frac{d}{dx}(\operatorname{coth}^{-1}x) &= \frac{1}{1-x^2}\end{aligned}$$

Student note

34. Find $\frac{d}{dx}(\tanh^{-1}(\sin x))$.

Student note

35. Find the exact value of each expression $\sinh(0)$, $\cosh(0)$, $\tanh(0)$, $\tanh(1)$, $\sinh(\ln 2)$, $\cosh(\ln 3)$, $\sinh^{-1}(1)$, $\cosh^{-1}(1)$.

Student note

36. Prove the identity: $\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$.

Student note

37. Prove the identity
 $\cosh(x) + \sinh(x) = e^x$
 $\sinh(2x) = 2 \sinh(x) \cosh(x)$

Student note

38. Find the derivative of the following functions

$$f_1(x) = x \cosh x, f_2(x) = \sinh^2 x.$$

Student note

39. Find the derivative of the following functions

$$f_3(x) = \sinh(\cosh x), f_4(x) = \tanh^{-1} \sqrt{x}$$

