2301107 Calculus I
4. Transcendental functions

Outline

4.1. Exponential function

4.2. Natural logarithmic function

4.3. Trigonometric function

4.4. Inverse trigonometric function

4.5. Hyperbolic function

4.1. Exponential function

- An exponential function is a function of the form

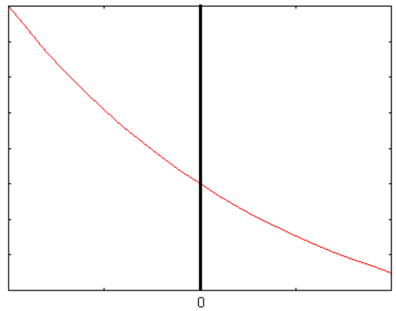
$$f(x) = a^x$$

  where a is a positive constant.
- If $x = n$, a positive integer, then $a^n = \underbrace{a \times a \times \cdots \times a}_{n\ \ terms}$
- If $x = 0$, then $a^0 = 1$.
- If $x = -n$, where n is a positive integer, then $a^{-n} = \dfrac{1}{a^n}$
- If x is a rational number, $x = p/q$ then

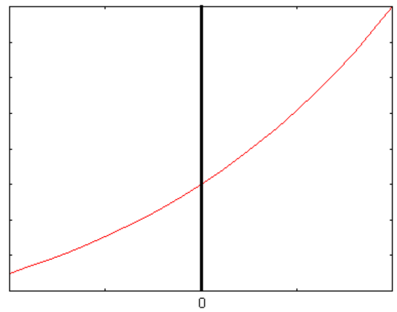
$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

- If x is an irrational number, $a^x = \lim_{r \to x} a^r, r \ \ rational$

Graph of the exponential function



$y = a^x$, $0 < a < 1$



$y = a^x, a > 1$

For $y = 1^x$, we have constant function $y = 1$.

Exponential function

- Theorem: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain = set of real number and range $(0, \infty)$.
- For $a, b > 0$ and any real numbers x, y then

$$a^{x+y} = a^x a^y \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x \qquad a^{x-y} = \frac{a^x}{a^y}$$

- Laws of Exponents:

  If $a > 1$ then $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to-\infty} a^x = 0$

  If $0 < a < 1$, then $\lim_{x\to\infty} a^x = 0$, $\lim_{x\to-\infty} a^x = \infty$

Derivative of exponential function

- For $a > 0$ and $a \neq 1$, the derivative of $f(x) = a^x$ is

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} \end{aligned}$$

- We have $f'(0) = \lim_{h\to 0}\frac{a^h-1}{h}$
- Therefore, $f'(x) = f'(0)a^x$
- The rate of change of any exponential function is proportional to the function itself.

Definition of the Number e

- Definition of the Number e:

  e is the number such that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.

- The derivative of the Natural exponential function

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

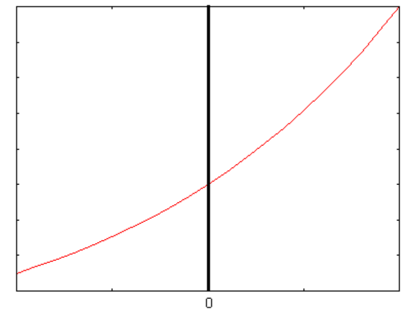
Student note

1. Differentiate the function $y = e^{\tan x}$.

Student note

2. Apply the chain rule where $u = f(x)$, $\dfrac{d}{dx}\left(e^{u}\right)=e^{u}\dfrac{du}{dx}$.

Exponential Graph



The exponential function $f(x) = e^x$, is an increasing continuous function on real number with the range $(0, \infty)$. Also,

$$\lim_{x\to-\infty} e^{x}=0, \lim_{x\to\infty} e^{x}=\infty.$$

Student note

3. Find the domain of each function

$$f_1(x)=\frac{1}{1+e^{x}} \quad f_2(x)=\frac{1}{1-e^{x}} \quad f_3(x)=\sqrt{1-2^{x}}$$

Student note

4. Differentiate the function

$$f_4(x)=x^2e^x, f_5(x)=\frac{e^x}{1+x}, f_6(x)=\sqrt{x}\,e^x.$$

Student note

5. Differentiate the function

$$f_7(u)=e^{\frac{1}{u}},\ f_8(x)=e^{k\tan\sqrt{x}},\ f_9(x)=\cos(e^{\pi x}).$$

Student note

6. Find y' if $e^{x^2 y}=x+y$.

Student note

7. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

Student note

8. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, $f'(0) = 2$.

4.2. Logarithmic function

- If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one. The inverse is called the logarithmic function with base a, denoted by $\log_a$.

$$f^{-1}(x) = y \leftrightarrow f(y) = x$$

$$\log_a x = y \leftrightarrow a^y = x$$

$$a^{\log_a x} = x$$

- Note that

$$\log_a(a^x) = x$$

Student note

9. Evaluate $\log_3 81$, $\log_{25} 5$ and $\log_{10} 0.0001$.

Logarithmic properties

- Theorem: If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one continuous, increasing function with domain $(0, \infty)$ and range is the set of real numbers.
- If $x, y > 0$ and r is a real number

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r \log_a x$$

- If $a > 1$, then $\lim_{x\to\infty} \log_a x = \infty$, $\lim_{x\to 0^+} \log_a x = -\infty$

Student note

10. Evaluate $\log_4 2 + \log_4 32$, $\log_2 80 - \log_2 5$.

Natural logarithm

- The logarithm with base e is called the natural logarithm and has a special notation
$$\log_e x = \ln x$$
- Properties
  - $\ln x = y \leftrightarrow e^y = x$
  - $\ln(e^x) = x$
  - $e^{\ln x} = x,\ x > 0$
  - $\ln e = 1$
- Change of base formula: For any positive number $a \neq 1$,
$$\log_a x = \frac{\ln x}{\ln a}$$

Student note

11. Use the properties of logarithms to expand
$$\log_2\left(\frac{x^3 y}{z^2}\right),\ \ln\sqrt{a(b^2+c^2)},\ \ln(uv)^{10},\ \ln\frac{3x^2}{(x+1)^5}$$

Student note

12. Solve each equation for x
    1) $2\ln x = 1$
    2) $e^{-x} = 5$
    3) $e^{3x+1} = k$
    4) $7e^x - e^{2x} = 12$

Student note

13. Find the domain and the inverse of the following functions
$f(x) = \log_2(4x - 2)$, $g(t) = \ln(e^t - 2)$.

Derivative of logarithmic function

- The derivative of the natural logarithm is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

- Properties

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, \frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$$

Student note

14. Differentiate $y = \ln(x^3 + 1)$, $f(x) = \ln(\sin x)$, $g(x) = \sqrt{\ln x}$.

Student note

15. Find $f'(x)$ if $f(x) = \ln |x|$.

Logarithmic differentiation

- We can simplify the derivatives of complicated functions involving products, quotients or powers using the logarithms, called **logarithmic differentiation.**
- Steps
  - Take logarithms of both sides, $y = f(x)$
  - Differentiate implicitly with respect to x.
  - Solve the resulting equation for y'.

- <u>The power rule</u>: If $f(x) = x^n$ then

$$f'(x) = n\, x^{n-1}.$$

Student note

16. Differentiate $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Logarithmic formula

$$\frac{d}{dx}\left(a^b\right) = 0$$
$$\frac{d}{dx}\left(f(x)\right)^b = b\left(f(x)\right)^{b-1} f'(x)$$
$$\frac{d}{dx}\left(a^{g(x)}\right) = a^{g(x)} \ln a \, g'(x)$$

- The number e as a limit,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Student note

17. Differentiate $y = x^{\sqrt{x}}$.

Student note

18. Differentiate the following functions

$f_1(x) = \ln(x^2+10)$ $\quad f_2(x) = \dfrac{1+\ln x}{1-\ln x}$ $\quad f_3(x) = \sqrt[5]{\ln x}$

Student note

19. Differentiate the following functions

$$f_4(x) = \cos(\ln x) \qquad f_5(x) = \sqrt{x}\ln x \qquad f_6(x) = \ln\sqrt[5]{x}$$

Student note

20. Find y' if $y = \ln(x^2 + y^2)$.

Student note

21. Find y' if $x^y = y^x$.

Integral of transcendental functions

- Consider the integration of an exponential function, because the exponential function $y = e^x$ is its derivative, therefore

$$\int e^x \, dx = e^x + C$$

Student note

22. Find the area under the curve $y = e^{-3x}$ from 0 to 1.

Student note

23. Evaluate $\int x^2 e^{x^3} dx$

Student note

24. Evaluate the integral $\int_0^5 e^{-3x} dx$, $\int_0^1 x e^{-x^2} dx$, $\int e^x \sqrt{1+e^x} dx$

Student note

25. Evaluate the integral $\int \frac{e^x+1}{e^x} dx$, $\int \frac{e^{\frac{1}{x}}}{x^2}$, $\int e^x \sin(e^x) dx$

Integral of logarithmic function

- Note that

$$\frac{d}{dx}\ln|x|=\frac{1}{x}$$

- Hence,

$$\int \frac{1}{x}dx=\ln|x|+C$$

- Properties

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a} \quad \frac{d}{dx}(a^x)=a^x\ln a \quad \int a^x\,dx=\frac{a^x}{\ln a}+C$$

Student note

26. Evaluate $\int \frac{x}{x^2+1}dx$, $\int_1^e \frac{\ln x}{x}dx$, $\int \tan x\,dx$

Student note

27. Evaluate the integral

$$\int_0^5 2^x\,dx,\ \int_2^4 \frac{3}{x}dx,\ \int_1^2 \frac{4+u^2}{u^3}du,\ \int_1^2 10^t\,dt,\ \int x\,2^{x^2}dx$$

Table of derivatives of inverse trigonometric function

$$\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccsc} x)=-\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arcsec} x)=\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arccot} x)=-\frac{1}{1+x^2}$$

Inverse Trigonometric function

- The two most useful integration formulae are

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$

Student note

28. Find $\displaystyle\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$

Student note

29. Evaluate $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx, \quad \int \frac{x}{x^4+9}\,dx$

Student note

30. Evaluate the integral

$$\int_0^1 \frac{4}{t^2+1}\,dt,\ \int \frac{1}{\sqrt{1-4t^2}}\,dt,\ \int \frac{\arctan x}{1+x^2}\,dx,\ \int \frac{t^2}{\sqrt{t-t^6}}\,dt$$

Hyperbolic function

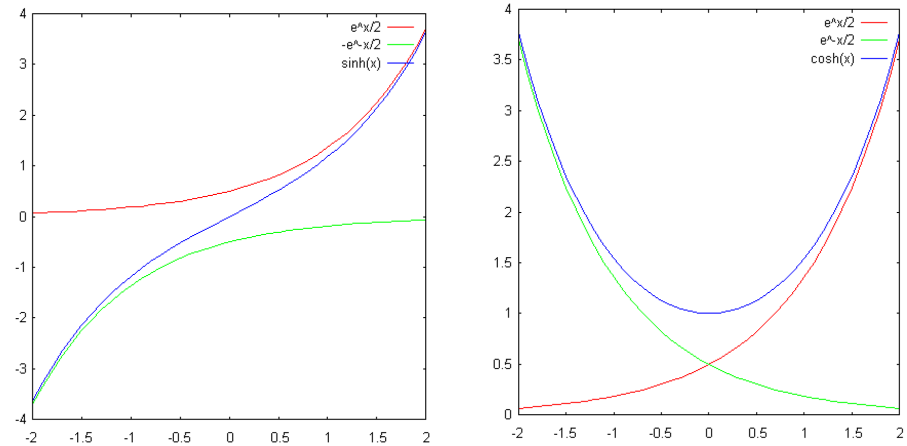
- Certain combination of the exponential functions e^x and e^{-x} arise so frequently in mathematics, which is called hyperbolic function.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Graph of Hyperbolic function



- Graph of $\sinh x$ and $\cosh x$.

Hyperbolic identities

- $\sinh(-x) = -\sinh(x)$ $\qquad \cosh(-x) = \cosh(x)$
- $\cosh^2 x - \sinh^2 x = 1$ $\qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

- Derivative of Hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Integrals of hyperbolic functions

$$\int \cosh x \, dx = \sinh x + c$$

$$\int \sinh x \, dx = \cosh x \, c$$

$$\int \tanh x \, dx = \operatorname{sech}^2 x + c$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \operatorname{arcsinh} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arccosh} x + C$$

$$\int \frac{1}{1-x^2} dx = \operatorname{arctanh} x + C$$

Student note

31. Evaluate the integral $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$ $\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt$

Student note

32. Evaluate the integral

$$\int \sinh x \cosh^2 x\,dx,\ \int \tanh x\,dx,\ \int \frac{\cosh x}{\cosh^2 x - 1}\,dt$$

Student note

33. Prove that $\cosh^2 x - \sinh^2 x = 1$ and $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Inverse hyperbolic functions

- The inverse of hyperbolic function

$$\sinh^{-1} x = y \leftrightarrow \sinh y = x,$$
$$\cosh^{-1} x = y \leftrightarrow \cosh y = x, \text{ and } y \geq 0$$
$$\tanh^{-1} x = y \leftrightarrow \tanh y = x.$$

- Formula for compute the inverse hyperbolic

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),\ x \geqslant 0$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Derivatives of Inverse hyperbolic functions

$$\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(csch^{-1}x)=-\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(sech^{-1}x)=-\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x)=\frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x)=\frac{1}{1-x^2}$$

Student note

34. Find $\frac{d}{dx}(\tanh^{-1}(\sin x))$.

Student note

35. Find the exact value of each expression sinh(0), cosh(0), tanh(0), tanh(1), sinh(ln 2), cosh(ln 3), $\sinh^{-1}(1)$, $\cos^{-1}(1)$.

Student note

36. Prove the identity: sinh(-x) = -sinh(x) and cosh(-x) = cosh(x).

Student note

37. Prove the identity

$\cosh(x) + \sinh(x) = e^x$

$\sinh(2x) = 2\sinh(x)\cosh(x)$

Student note

38. Find the derivative of the following functions

$f_1(x) = x\cosh x$, $f_2(x) = \sinh^2 x$.

Student note

39. Find the derivative of the following functions

$f_3(x) = \sinh(\cosh x)$, $f_4(x) = \tanh^{-1}\sqrt{x}$