2301107 Calculus I

5. Applications of differentiation

Outline

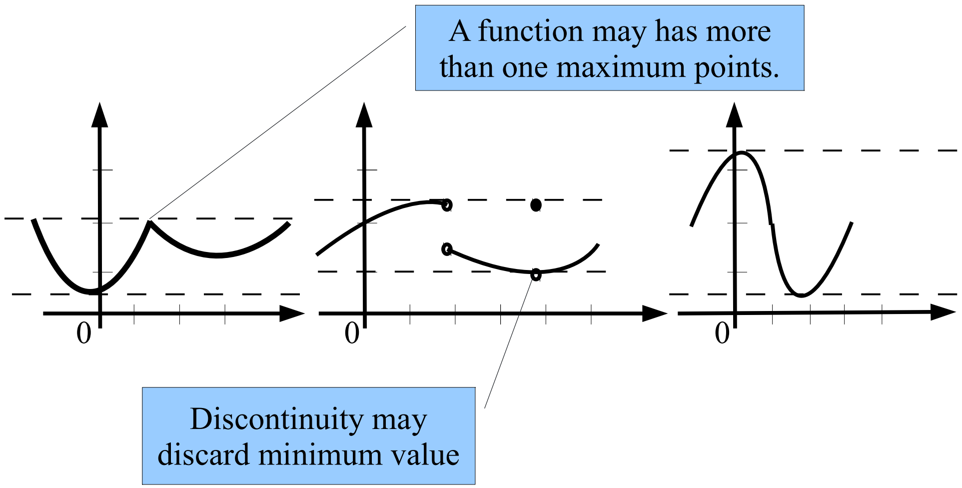
5.1. Extreme values

- Optimization problems (determine the optimal way of doing something) is the important application of differential calculus such as
  - What is the shape of a can that minimizes manufacturing costs?
  - What is the maximum acceleration of a space shuttle?
  - At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

Maximum and minimum values

- Definition: A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in the domain of f.
  - The number $f(c)$ is called the **maximum value** of f.
- Similarly, f has an **absolute minimum** (or **global minimum**) at d if $f(d) \leq f(x)$ for all x in the domain of f.
  - The number $f(d)$ is called the **minimum value** of f.
- The maximum and minimum values of f are called the **extreme values** of f.

Maximum and minimum



Maximum and minimum values

- Definition: A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c (for all x in some open interval containing c).
- Similarly, f has an **local minimum** (or **relative minimum**) at d if $f(d) \leq f(x)$ when x is near d (for all x in some open interval containing d).

Student note

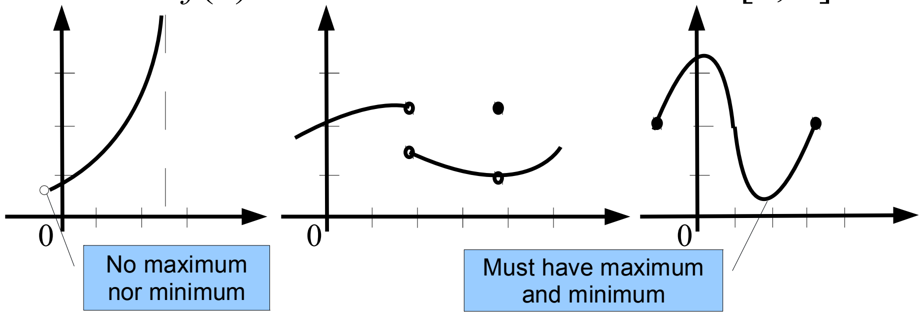
1. Determine the minimum and maximum values for $f(x) = \cos(x)$.

Student note

2. Graph $f(x) = x^2$ and $f(x) = x^3$ and determine the minimum and maximum values if exist.

The Extreme Value Theorem

- The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value of $f(d)$ at some numbers c and d in $[a, b]$.



Fermat's Theorem

- Fermat's Theorem: If f has a local maximum at c, and if $f'(c)$ exists, then $f'(c) = 0$.
- Proof: $f(c) \geq f(c + h)$ $\quad \dfrac{f(c+h)-f(c)}{h} \leq 0$

$$\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

- Hence, $f'(c) \leq 0$.
- Similarly, $\dfrac{f(c+h)-f(c)}{h} \geq 0$, $\displaystyle\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$

Student note

3. If $f(x) = x^3$, determine the maximum value of f if exists.

Student note

4. If $g(x) = |x|$, determine the maximum and minimum value of g if exist.

Critical number

- <u>Definition</u>: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
- If f has a local maximum or minimum at c then c is a critical number of f.

Student note

5. Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Student note

6. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1, \qquad -\tfrac{1}{2} \le x \le 4.$$

The Closed Interval Method

- To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$
  1. Find the values of f at the critical numbers of f in (a, b)
  2. Find the values of f at the endpoints of the interval.
  3. The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Student note

7. Find the critical numbers of the following functions.

$(a)\quad g(x)=|3x-2| \quad (b)\quad f(\theta)=2\cos\theta+\sin^2\theta$

$(c)\quad h(t)=\sqrt{t}(1-t) \quad (d)\quad g(\theta)=2\theta-\tan\theta$

Optimization problems

- Finding the extreme values having many applications
  - A traveler may want to minimize transporation time
  - Fermat's Principle in optics states that light follows the path that takes the least time.
  - A businessperson may want to minimize costs and maximize profit
- Solving a practical problem is required to convert the word problem into a mathematical optimization problem.

Steps in solving optimizations

- Understand the problem; What is the unknown? What are the given quantities and conditions?
- Draw a diagram; To see interactions among quantities
- Introduce notation; Assign a symbol, *Q,* to be maximized or minimized and other variables
- Express *Q* in terms of other symbols
- Simplify the expression, until $Q = f(x)$
- Use methods from Calculus to find the *absolute* maximum or minimum of *f*.

Student note

8. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Student note

9. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

First Derivative Test for Absolute Extreme Value

- Suppose that c is a critical number of a continuous function f defined on an interval.
- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Student note

10. Find the point on the parabola $y^2 = 2x$ that is closet to the point (1, -4).

Student note

11. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Student note

12. Find the dimensions of a rectangle with area 1000 m^2 whose parameter is as small as possible.

Student note

13. Find the positive numbers whose product is 100 and whose sum is a minimum.

Applications to Business and Economics

- For the **cost function** $C(x)$, we define the **marginal cost** as the rate of change of C with respect to x.
- The **average cost function** is defined as

$$c(x) = C(x)/x$$

- If the average cost is a minimum, then

  marginal cost = average cost

- Given a **demand function** $p(x)$ (the price per unit that the company can charge if it sells x units), we called

$$R(x) = xp(x)$$

  the **revenue function** and the derivative of R is called the **marginal revenue function**.

Student note

14. A company estimates that the cost (in dollars) of producing x items is $C(x) = 2600 + 2x + 0.001x^2$.
    a) Find the cost, average cost, and marginal cost of producing 1000 items, 2000 items and 3000 items.
    b) At what production level will the average cost be lowest, and what is the minimum average cost?

Student note

15. Determine the production level that will maximize the profit for a company with cost and demand functions

$$C(x) = 84 + 1.26x - 0.01x^2 + 0.00007x^3$$

and $p(x) = 3.5 - 0.01x$

Applications to business and economics

- If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function. If

$$P'(x) = R'(x) - C'(x) = 0$$

then $R'(x) = C'(x)$

- If the profit is a maximum, then

marginal revenue = marginal cost

Using the second derivative, the profit will be maximum when $R'(x) = C'(x)$ and $R''(x) < C''(x)$.

Student note

16. A store has been selling 200 DVD players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of players sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Student note

17. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

5.2. How derivatives affect the shape of a graph

- The derivative of $y = f(x)$ can tell us where a function is increasing or decreasing.
- **Increasing/Decreasing Test**:

(a) If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.

(a) If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

Student notes

18. Find where the function

$$f(x) = 3x^3 - 6x^2 + x$$

is increasing and where it is decreasing.

The First Derivative Test

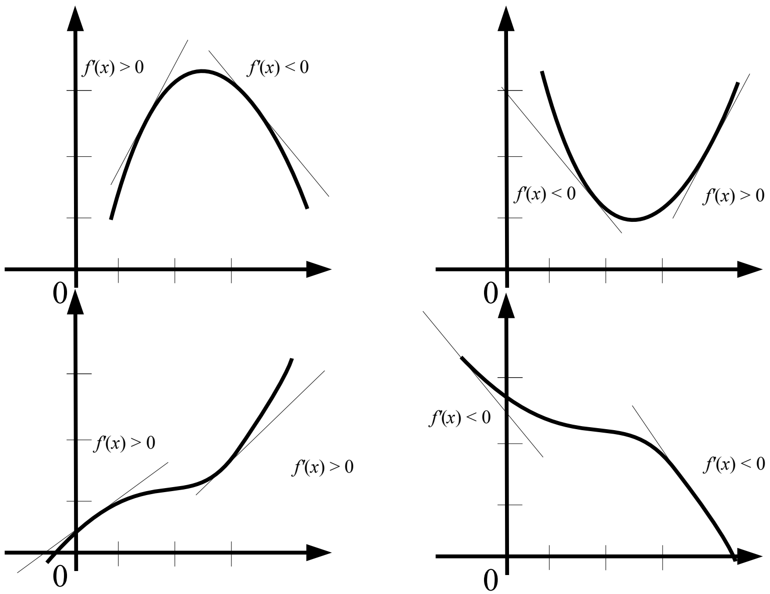
- Suppose that c is a critical number of a continuous function f.

(a) If $f\,'$ changes from positive to negative at c, then f has a local maximum at c.

(b) If $f\,'$ changes from negative to positive at c, then f has a local minimum at c.

(c) If $f\,'$ does not change sign at c (such as $f\,'$ is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

The First Derivative Test



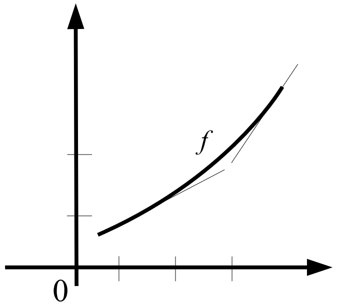
Student note

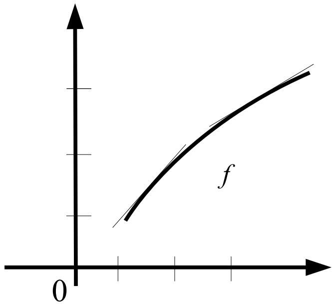
19. Find the local maximum and minimum values of the function

$$g(x) = x - 2 \sin x, \quad 0 \le x \le 2\pi.$$

Concave upward/downward

- Definition:If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f. lies below all of its tangents on I, it is called **concave downward** on I.





Concavity test

- Concavity test:

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

- Definition: A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Student note

20. Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f(0) = 0, f(2) = 3, f(4) = 6$;

(ii) $f'(0) = f'(4) = 0$;

(iii) $f'(x) > 0$ for $0 < x < 4, f'(x) < 0$ for $x < 0$ and for $x > 4$

(iv) $f''(x) > 0$ for $x < 2, f''(x) < 0$ for $x > 2$.

The Second Derivative Test

- The Second Derivative Test: Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Student note

21. Discuss the following curve and determines concavity, points of inflection, local maxima and local minima. Then sketch the curve

$$y = x^4 - 2x^3$$

Student note

22. Sketch the curve $f(x) = x + 2\cos x,\ \ 0 \le x \le 2\pi$.

Student note

23. Find a formula for a function f that satisfies the following conditions:

$(a)\ \lim_{x\to\pm\infty} f(x)=0, \lim_{x\to 0} f(x)=-\infty, f(2)=0$

$(b)\ \lim_{x\to 3^-} f(x)=\infty, \lim_{x\to 3^+} f(x)=-\infty$

5.3. Curve sketching

- Domain: Start from identifying the set of x for which $f(x)$ is defined.
- Intercepts: The y-intercept is $f(0)$ and the x-intercept is when we set $y = 0$ and solve for x.
- Symmetry: $f(-x) = f(x)$ or even function, the curve is symmetric about the y-axis; $f(-x) = -f(x)$ or odd function, the curve is symmetric about the origin.
- Asymptotes: Determine horizontal asymptotes and vertical asymptotes $\lim_{x\to\infty} f(x)=L,\ \lim_{x\to-\infty} f(x)=L,$

$$\lim_{x\to a} f(x)=\infty,\ \lim_{x\to b} f(x)=-\infty$$

Curve sketching

- Intervals of Increase or Decrease: Determine the intervals on which $f'(x)$ is positive (increasing) and $f'(x)$ is negative (decreasing)
- Local Maximum and Minimum values: Find the critical numbers of f and determine the local minimum or the local maximum
- Concavity and points of inflection: Compute $f''(x)$ and use the Concavity test.
- Sketch the Curve: Using information from above and do the sketch

Student note

24. Sketch the curve $y=\dfrac{2x^2}{x^2-1}$.

Student note

25. Sketch the curve $f(x)=\dfrac{x^2}{\sqrt{x+1}}$

Student note

26. Sketch the graph of $f(x) = 2 \cos x + \sin 2x$.

Student note

27. Sketch the curve $f(x) = x - 3x^{1/3}$.

Student note

28. Sketch the curve $f(x)=\sqrt{\dfrac{x}{x-5}}$

Student note

29. Sketch the curve $f(x)=\sqrt{x^2+1}-x$

Rates of change

- Suppose y depends on x, $y = f(x)$.
- The change in x from x_1 to x_2 is called **the increment** of x is $\Delta x = x_2 - x_1$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

- The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change** of y with respect to x over the interval $[x_1, x_2]$.

Rates of change

- Instantaneous rate of change

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Student note

30. Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m. above the ground. What is the velocity of the ball after 5 seconds? How fast is the ball traveling when it hits the ground?

Student note

31. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3+6t+2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$ and $t = 3$.

Student note

32. Find an equation of the tangent line to the curve at the given point.

1. $y=1+2x-x^3, (1,2)$
2. $y=\sqrt{2x+1}, (4,3)$
3. $y=\dfrac{x-1}{x-2}, (3,2)$
4. $y=\dfrac{2x}{(x+1)^2}, (0,0)$

Student note

33. Find the slope of the tangent to the parabola $y = 1 + x + x^2$ at the point where $x = a$. Find the slopes of the tangent lines at the points whose x-coordinate are (i) -1, (ii) -½, (iii) 1.

Student note

34. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

5.4. Related rates

- Note that if we are pumping air into a balloon, both the volume and radius are increasing and their rates of increase are related to each other.
- In related rates problem, we want to compute the related rate of one quantity in terms of the rate of change of another quantity (may be easily measured).
- Procedure:
  - Determine the equation that relates two quantities
  - Use the Chain Rule to differentiate both sides
  - Rewrite one related rate in term of others.

Student notes

35. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm.?

Student notes

36. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Student notes

37. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Student notes

38. Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection.

Student notes

39. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closet to the searchlight?

Student notes

40. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 P.M.?

Student notes

41. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

5.5. L'Hospital's Rule

- Some limit of function, we cannot apply the regular law of limits, for example,

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

- Note that the limit of both numerator and denominator are 0 and $\frac{0}{0}$ is not defined.

We called this limit as **indeterminate form of type** $\frac{0}{0}$

Indeterminate form

- Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote

$$\lim_{x \to \infty} \frac{\ln x}{x-1}$$

- Note that the limit of both numerator and denominator are ∞ and $\frac{\infty}{\infty}$ is not defined.

We called this limit as **indeterminate form of type $\frac{\infty}{\infty}$**

L'Hospital's Rule

- Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

or that

$\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists.

L'Hospital's Rule

- Note 1: Before using the L'Hospital's Rule, the given conditions must satisfied.
- Note 2: L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity
- Note 3: For the special case in which $f(a)=g(a)=0$, f' and g' are continuous, and $g'(a) \neq 0$. It can be shown that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hospital's Rule

$$\begin{aligned}
\lim_{x \to a} \frac{f'(x)}{g'(x)} &= \frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim\limits_{x \to a} \frac{g(x)-g(a)}{x-a}} \\
&= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
&= \lim_{x \to a} \frac{f(x)}{g(x)}
\end{aligned}$$

Student notes

42. Find $\lim\limits_{x\to 1}\dfrac{\ln(x)}{x-1}, \lim\limits_{x\to\infty}\dfrac{e^x}{x^2}$

Student notes

43. Find $\lim\limits_{x\to\infty}\dfrac{\ln(x)}{\sqrt[3]{x}}, \lim\limits_{x\to 0}\dfrac{\tan(x)-x}{x^3}$

Student notes

44. Evaluate $\lim_{x \to 0^+} x \ln x$, $\lim_{x \to \frac{\pi}{2}^-} (sec\, x - \tan x)$

Student notes

45. Evaluate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

Student notes

46. Evaluate $\lim_{x \to 0^+} x^x$

Student notes

47. Evaluate $\lim_{t \to 0} \frac{e^t - 1}{t^3}, \lim_{x \to 0} \frac{x + \tan x}{\sin x}, \lim_{x \to 1} \frac{x^a - a x + a - 1}{(x-1)^2}$

Student notes

48. Evaluate $\lim_{x\to 0}\dfrac{x}{\arctan(4x)}, \lim_{x\to\infty} x^{\frac{1}{x}}, \lim_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{x}$

Student notes

49. Evaluate $\lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx}, \lim_{x\to 0^{+}}(\tan 2x)^{x}, \lim_{x\to 0^{+}}(\cos x)^{\frac{1}{x^{2}}}$

Student notes

50. If f' is continuous, $f(2) = 0$ and $f'(2) = 8$, evaluate

$$\lim_{x \to 0} \frac{f(2+3x)+f(2+5x)}{x}$$