2301107 Calculus I
6. Techniques of Integration

Outline

6.1. Integration by parts

6.2. Integration of rational functions by partial fractions

6.3. Integration of trigonometric functions

6.4. Trigonometric substitution

Integral formulae

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C \qquad \int \frac{1}{x}\,dx=\ln|x|+C$$

$$\int e^x\,dx=e^x+C \qquad \int a^x\,dx=\frac{a^x}{\ln a}+C$$

$$\int \sin x\,dx=-\cos x+C \qquad \int \cos x\,dx=\sin x+C$$

$$\int \sec^2 x\,dx=\tan x+C \qquad \int \csc^2 x\,dx=-\cot x+C$$

$$\int \sec x\tan x\,dx=\sec x+C \qquad \int \csc x\cot x\,dx=-\csc x+C$$

$$\int \tan x\,dx=\ln|\sec x|+C \qquad \int \cot x\,dx=\ln|\sin x|+C$$

$$\int \sinh x\,dx=\cosh x+C \qquad \int \cosh x\,dx=\sinh x+C$$

$$\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\arcsin\left(\frac{x}{a}\right)+C$$

6.1. Integration by parts

- Differentiation of the product *f* and *g*

$$\frac{d}{dx}\left(f(x)g(x)\right)=f(x)g'(x)+g(x)f'(x)$$

$$d\left(f(x)g(x)\right)=f(x)g'(x)dx+g(x)f'(x)dx$$

Then,

$$f(x)g(x)=\int f(x)g'(x)dx+\int g(x)f'(x)dx$$

$$\int f(x)g'(x)dx=f(x)g(x)-\int g(x)f'(x)dx$$

Formula for integration by parts

- The formula for integration by parts:

$$\int u\, dv = uv - \int v\, du$$

- For the definite integral:

$$\int_a^b f(x)\,g'(x)\,dx = \big(f(b)\,g(b) - f(a)\,g(a)\big) - \int_a^b g(x)\,f'(x)\,dx$$

Student note

1. Find $\int x \sin x \,\mathrm{d}x$, $\int \ln x \,\mathrm{d}x$

Student note

2. Find $\int t^2 e^t \, dt$, $\int e^x \sin x \, dx$

Student note

3. Evaluate $\int_0^1 \arctan x \, dx$

Student note

4. Prove the reduction formula

$$\int sin^n x\,dx = -\frac{1}{n} cos\,x\,sin^{n-1} x + \frac{n-1}{n} \int sin^{n-2} x\,dx$$

Student note

5. Use the reduction formula to show that

$$\int_0^{\frac{\pi}{2}} sin^n x\,dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} sin^{n-2} x\,dx$$

6.2. Integration of rational functions by partial fractions I

- A rational function is a function of a ratio of polynomials.
- We simplify the rational function by rewriting as a sum of simpler fractions, called partial fractions. For example,

$$\frac{2}{x-1}-\frac{1}{x+2}=\frac{2(x+2)-(x-1)}{(x-1)(x+2)}=\frac{x+5}{x^2+x-2}$$

That means

$$\int\frac{x+5}{x^2+x-2}dx=\int\frac{2}{x-1}-\frac{1}{x+2}dx=2\ln|x-1|-\ln|x+2|+C$$

Integration of rational functions by Partial fractions II

- Given a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

- If $\deg(P) \geq \deg(Q)$, divide by $Q(x)$, then a remainder $R(x)$ has smaller degree $\deg(R) < \deg(Q)$.

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

Student note

6. Find $\int \frac{x^3+x}{x-1}dx$

CASE I

- The denominator $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Student note

7. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

Student note

8. Find $\int \frac{1}{x^2-a^2}dx$

CASE II

- The denominator $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where some $a_ix + b_i$ are repeated r times. Then we have r terms of

$$\frac{A_1}{a_i x+b_i}+\frac{A_2}{(a_i x+b_i)^2}+\cdots+\frac{A_r}{(a_i x+b_i)^r}$$

Student note

9. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

CASE III

- The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated. Then

$$\frac{Ax+b}{ax^2+bx+c}$$

- We also use the formula

$$\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

Student note

10. Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Student note

11. Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3}dx$

CASE IV

- The denominator $Q(x)$ contains a repeated irreducible quadratic factors. For each repeated quadratic factor that is not reducible with r terms,

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Student note

12. Write out the partial fraction decomposition of the function $\dfrac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

Student note

13. Evaluate $\int \dfrac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$

Rationalizing substitution

- Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, an expression of the form $u = \sqrt[n]{g(x)}$

Student note

14. Evaluate $\int \frac{\sqrt{x+4}}{x} dx$

6.3. Trigonometric integrals I

- Strategy for evaluating $\int sin^m x\ cos^n x\ dx$
  - If the power of cosine is odd, save one cosine factor and use $cos^2 x = 1 - sin^2 x$, then substitute $u = sin\ x$.
  - If the power of sine is odd, save one sine factor and use $sin^2 x = 1 - cos^2 x$, then substitute $u = cos\ x$
  - Otherwise, use the half angle identities

$$sin^2 x = (1 - cos\ 2x)/2,\ cos^2 x = (1 + cos\ 2x)/2$$

$$\text{or } \sin x \cos x = (sin\ 2x)/2$$

Student note

15. Evaluate $\int cos^3 x\ dx$, $\int sin^5 x\ cos^2 x\ dx$, $\int sin^4 x\ dx$

Trigonometric integrals II

- Strategy for evaluating $\int tan^m x \; sec^n x \; dx$
  - If the power of secant is even and $n > 1$, save a factor of $sec^2 x$, use $sec^2 x = 1 + tan^2 x$, then substitute $u = tan\, x$.
  - If the power of tangent is odd, save a factor of $sec(x)\, tan(x)$ and use $tan^2 x = sec^2 x - 1$, substitute $u = sec\, x$.
- Other case, we may need

$$\int tan\, x \; dx = \ln | sec\, x | + C$$

$$\int sec\, x \; dx = \ln | sec\, x + tan\, x | + C$$

Student note

16. Find $\int tan^3 x \; dx$, $\int sec^3 x \; dx$.

Student note

17. Evaluate $\int tan^6x\ sec^4x\ \mathrm{d}x$, $\int tan^5x\ sec^7x\ \mathrm{d}x$.

Trigonometric integrals III

- Strategy for evaluating $\int sin\,mx\ cos\ nx\ \mathrm{d}x$, $\int sin\,mx\ sin\ nx\ \mathrm{d}x$, $\int cos\,mx\ cos\ nx\ \mathrm{d}x$ use the corresponding identity:

$$sin\,A\,cos\,B=\frac{sin(A-B)+sin(A+B)}{2}$$

$$sin\,A\,sin\,B=\frac{cos(A-B)-cos(A+B)}{2}$$

$$cos\,A\,cos\,B=\frac{cos(A-B)+cos(A+B)}{2}$$

Student note

18. Find $\int \sin 4x \cos 5x \, \mathrm{d}x$.

Student note

19. Find the area of the region bounded by
    1. $y = \sin x$, $y = \sin^3 x$, $x = 0$, $x = \pi/2$
    2. $y = \sin x$, $y = 2 \sin^2 x$, $x = 0$, $x = \pi/2$

Student note

20. Prove the formula where m and n are positive integers.

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{otherwise} \end{cases}$$

Student note

21. Prove that the area of a circle with radius r is πr^2.

6.4. Trigonometric substitution

- In finding the area of a circle, we face with the integral of the form $\int \sqrt{a^2 - x^2}\,dx$ where $a > 0$.
- To get rid of the square root sign, we substitute x by $a \sin \theta$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a| \sqrt{\cos^2 \theta} = a |\cos \theta|$$

Student note

22. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$

Student note

23. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Table of Trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta, -\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta, 0\leqslant\theta<\frac{\pi}{2}\vee\pi\leqslant\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Student note

24. Find $\int \frac{1}{x^2\sqrt{x^2+4}}dx$, $\int \frac{x}{\sqrt{x^2+4}}dx$, $\int_0^{3\frac{\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}dx$

Student note

25. Evaluate $\int \frac{1}{\sqrt{t^2-6t+13}}dt$, $\int x\sqrt{1-x^4}dx$

Student note

26. Evaluate $\int \frac{\sqrt{1+x^2}}{x}dx$, $\int_0^1 \sqrt{x^2+1}\,dx$

Student note

27. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

Strategy for integration

- Four-step strategy
  - Simplify the integrand if Possible
    $\sqrt{x}(1+\sqrt{x})=\sqrt{x}+x$, $\dfrac{\tan\theta}{sec^2\theta}=\sin\theta\cos\theta=\dfrac{1}{2}\sin 2\theta$
  - Look for an Obvious Substitution
    $$\int \frac{x}{x^2-1}dx\,,\, u=x^2-1$$
  - Classify the Integrand According to Its Form
    - Trigonometric, Rational, Integration by parts, Radicals
  - Try Again: Try substitution, Try integration by parts, Manipulate the integrand, etc.

Student note

28. Find $\int \dfrac{\tan^2 x}{\cos^3 x}dx\,,\int e^{\sqrt{x}}dx\,,\int \dfrac{x^5+1}{x^3-3x^2-10x}dx$

Student note

29. Find $\int \frac{1}{x\sqrt{\ln x}}\,dx$, $\int \sqrt{\frac{1-x}{1+x}}\,dx$, $\int_{-2}^{2} |x^2-4x|\,dx$

Student note

39. Find $\int_0^3 \frac{1}{x\sqrt{x}}\,dx$, $\int_0^1 \frac{1}{4y-1}\,dy$