2301107 Calculus I

6. Techniques of Integration

Outline

6.1. Integration by parts

6.2. Integration of rational functions by partial fractions

6.3. Integration of trigonometric functions

6.4. Trigonometric substitution

Integral formulae

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C \qquad \int \frac{1}{x}\,dx=\ln|x|+C$$

$$\int e^x\,dx=e^x+C \qquad \int a^x\,dx=\frac{a^x}{\ln a}+C$$

$$\int \sin x\,dx=-\cos x+C \qquad \int \cos x\,dx=\sin x+C$$

$$\int \sec^2 x\,dx=\tan x+C \qquad \int \csc^2 x\,dx=-\cot x+C$$

$$\int \sec x\tan x\,dx=\sec x+C \qquad \int \csc x\cot x\,dx=-\csc x+C$$

$$\int \tan x\,dx=\ln|\sec x|+C \qquad \int \cot x\,dx=\ln|\sin x|+C$$

$$\int \sinh x\,dx=\cosh x+C \qquad \int \cosh x\,dx=\sinh x+C$$

$$\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\arcsin\left(\frac{x}{a}\right)+C$$

6.1. Integration by parts

- Differentiation of the product f and g

$$\frac{d}{dx}\big(f(x)g(x)\big)=f(x)g'(x)+g(x)f'(x)$$

$$d\big(f(x)g(x)\big)=f(x)g'(x)\,dx+g(x)f'(x)\,dx$$

Then,

$$f(x)g(x)=\int f(x)g'(x)\,dx+\int g(x)f'(x)\,dx$$

$$\int f(x)g'(x)\,dx=f(x)g(x)-\int g(x)f'(x)\,dx$$

Formula for integration by parts

- The formula for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

- For the definite integral:

$$\int_a^b f(x)\,g'(x)\,dx = \left(f(b)\,g(b) - f(a)\,g(a)\right) - \int_a^b g(x)\,f'(x)\,dx$$

Student note

1. Find $\int x \sin x\, dx$, $\int \ln x\, dx$

Student note

2. Find $\int t^2 e^t\, dt$, $\int e^x \sin x\, dx$

Student note

3. Evaluate $\int_0^1 \arctan x\, dx$

Student note

4. Prove the reduction formula

$$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

Student note

5. Use the reduction formula to show that

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx$$

6.2. Integration of rational functions by partial fractions I

- A rational function is a function of a ratio of polynomials.
- We simplify the rational function by rewriting as a sum of simpler fractions, called partial fractions. For example,

$$\frac{2}{x-1}-\frac{1}{x+2}=\frac{2(x+2)-(x-1)}{(x-1)(x+2)}=\frac{x+5}{x^2+x-2}$$

That means

$$\int \frac{x+5}{x^2+x-2}dx=\int \frac{2}{x-1}-\frac{1}{x+2}dx=2\ln|x-1|-\ln|x+2|+C$$

Integration of rational functions by Partial fractions II

- Given a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are polynomials.
- If $\deg(P) \geq \deg(Q)$, divide by $Q(x)$, then a remainder $R(x)$ has smaller degree $\deg(R) < \deg(Q)$.

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

Student note

6. Find $\int \frac{x^3+x}{x-1}\,dx$

CASE I

- The denominator $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Student note

7. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Student note

8. Find $\int \frac{1}{x^2-a^2}\,dx$

CASE II

- The denominator $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

where some $a_ix + b_i$ are repeated r times. Then we have r terms of

$$\frac{A_1}{a_i x+b_i}+\frac{A_2}{(a_i x+b_i)^2}+\cdots+\frac{A_r}{(a_i x+b_i)^r}$$

Student note

9. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

CASE III

- The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated. Then

$$\frac{Ax+b}{ax^2+bx+c}$$

- We also use the formula

$$\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

Student note

10. Evaluate $\int \frac{2x^2-x+4}{x^3+4x}dx$

Student note

11. Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3}dx$

CASE IV

- The denominator $Q(x)$ contains a repeated irreducible quadratic factors. For each repeated quadratic factor that is not reducible with r terms,

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Student note

12. Write out the partial fraction decomposition of the function $\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

Student note

13. Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$

Rationalizing substitution

- Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, an expression of the form $u = \sqrt[n]{g(x)}$

Student note

14. Evaluate $\int \frac{\sqrt{x+4}}{x} dx$

6.3. Trigonometric integrals I

- Strategy for evaluating $\int \sin^m x \cos^n x \, dx$
  - If the power of cosine is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$, then substitute $u = \sin x$.
  - If the power of sine is odd, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$, then substitute $u = \cos x$
  - Otherwise, use the half angle identities

$$\sin^2 x = (1 - \cos 2x)/2, \cos^2 x = (1 + \cos 2x)/2$$

$$\text{or } \sin x \cos x = (\sin 2x)/2$$

Student note

15. Evaluate $\int \cos^3 x \, dx$, $\int \sin^5 x \cos^2 x \, dx$, $\int \sin^4 x \, dx$

Trigonometric integrals II

- Strategy for evaluating $\int tan^m x\ sec^n x\ dx$
  - If the power of secant is even and $n > 1$, save a factor of $sec^2 x$, use $sec^2 x = 1 + tan^2 x$, then substitute $u = tan\ x$.
  - If the power of tangent is odd, save a factor of $sec(x)\ tan(x)$ and use $tan^2 x = sec^2 x - 1$, substitute $u = sec\ x$.
- Other case, we may need

$$\int tan\ x\ dx = \ln|sec\ x| + C$$

$$\int sec\ x\ dx = \ln|sec\ x + tan\ x| + C$$

Student note

16. Find $\int tan^3 x\ dx$, $\int sec^3 x\ dx$.

Student note

17. Evaluate $\int tan^6 x\ sec^4 x\ dx$, $\int tan^5 x\ sec^7 x\ dx$.

Trigonometric integrals III

- Strategy for evaluating $\int sin\ mx\ cos\ nx\ dx$, $\int sin\ mx\ sin\ nx\ dx$, $\int cos\ mx\ cos\ nx\ dx$ use the corresponding identity:

$$sin\ A\ cos\ B = \frac{sin(A-B) + sin(A+B)}{2}$$

$$sin\ A\ sin\ B = \frac{cos(A-B) - cos(A+B)}{2}$$

$$cos\ A\ cos\ B = \frac{cos(A-B) + cos(A+B)}{2}$$

Student note

18. Find $\int \sin 4x \cos 5x \, dx$.

Student note

19. Find the area of the region bounded by
    1. $y = \sin x$, $y = \sin^3 x$, $x = 0$, $x = \pi/2$
    2. $y = \sin x$, $y = 2 \sin^2 x$, $x = 0$, $x = \pi/2$

Student note

20. Prove the formula where m and n are positive integers.

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{otherwise} \end{cases}$$

Student note

21. Prove that the area of a circle with radius r is πr^2.

6.4. Trigonometric substitution

- In finding the area of a circle, we face with the integral of the form $\int \sqrt{a^2 - x^2}\,dx$ where $a > 0$.
- To get rid of the square root sign, we substitute x by $a \sin \theta$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = |a| \sqrt{\cos^2 \theta} = a |\cos \theta|$$

Student note

22. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2} dx$

Student note

23. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Table of Trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, 0 \leqslant \theta < \frac{\pi}{2} \vee \pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Student note

24. Find $\int \frac{1}{x^2\sqrt{x^2+4}}dx$, $\int \frac{x}{\sqrt{x^2+4}}dx$, $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}dx$

Student note

25. Evaluate $\int \frac{1}{\sqrt{t^2-6t+13}}dt$, $\int x\sqrt{1-x^4}dx$

Student note

26. Evaluate $\int \frac{\sqrt{1+x^2}}{x}dx$, $\int_0^1 \sqrt{x^2+1}\,dx$

Student note

27. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

Strategy for integration

- Four-step strategy
  - Simplify the integrand if Possible
    $\sqrt{x}(1+\sqrt{x})=\sqrt{x}+x, \frac{\tan\theta}{sec^2\theta}=\sin\theta\cos\theta=\frac{1}{2}\sin 2\theta$
  - Look for an Obvious Substitution
    $\int\frac{x}{x^2-1}dx, u=x^2-1$
  - Classify the Integrand According to Its Form
    - Trigonometric, Rational, Integration by parts, Radicals
  - Try Again: Try substitution, Try integration by parts, Manipulate the integrand, etc.

Student note

28. Find $\int\frac{\tan^2 x}{\cos^3 x}dx, \int e^{\sqrt{x}}dx, \int\frac{x^5+1}{x^3-3x^2-10x}dx$

Student note

29. Find $\int\frac{1}{x\sqrt{\ln x}}dx, \int\sqrt{\frac{1-x}{1+x}}dx, \int_{-2}^{2}|x^2-4x|dx$

Student note

39. Find $\int_0^3\frac{1}{x\sqrt{x}}dx, \int_0^1\frac{1}{4y-1}dy$