

2301107 Calculus I

8. Improper Integrals

Chapter 8: Improper integrals

C08-2

Outline

8.1. Improper integral of type I

8.2. Improper integral of type II

8.3. Improper integral of mixed type

Integral formulae

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \tan x dx = \ln|\sec x| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$$

Improper integrals

- Sometime a function may not be continuous on $[a, b]$, or a and/or b are not finite. We also interested in

$$\int_a^b f(x) dx$$

for example, $\int_0^1 \frac{1}{x} dx$, $\int_{-\infty}^0 x e^x dx$, $\int_0^3 \frac{1}{x-1} dx$

8.1. Improper integrals of Type 1

- **Definition** of an **improper integral of Type 1**

– If $\int_a^t f(x) dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

– If $\int_t^b f(x) dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

Improper integrals of Type 1

- The improper integrals are convergent if the corresponding limit exists and divergent if the limit does not exist.

- If both $\int_a^\infty f(x) dx$, $\int_{-\infty}^a f(x) dx$ are convergent then

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

Student note

1. Determine whether the integral is convergent or divergent.

$$\int_1^{\infty} \frac{1}{x} dx, \int_1^{\infty} \frac{1}{x^2} dx, \int_{-\infty}^0 x e^x dx$$

Student note

2. Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx, \int_1^{\infty} \frac{1}{x^p} dx$$

8.2. Improper integrals of Type 2

- If f is continuous on $[a, b)$ and is discontinuous at b ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

- If f is continuous on $(a, b]$ and is discontinuous at a ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

- If f has a discontinuity at c , where $a < c < b$, and

both $\int_a^c f(x) dx$, $\int_c^b f(x) dx$ are convergent, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Student note

- Determine whether the integral is convergent or divergent.

$$\int_2^5 \frac{1}{\sqrt{x-2}} dx, \int_0^{\frac{\pi}{2}} \sec x dx$$

Student note

4. Determine whether the integral is convergent or divergent.

$$\int_0^3 \frac{1}{x-1} dx, \int_0^1 \ln x dx$$

A comparison test for improper integrals

- Suppose that f and g are continuous functions with $f(x) > g(x) > 0$ for $x \geq a$,
 - If $\int_a^\infty f(x) dx$ is convergent, then $\int_a^\infty g(x) dx$ is convergent
 - If $\int_a^\infty g(x) dx$ is divergent, then $\int_a^\infty f(x) dx$ is divergent.

Student note

5. Show that $\int_0^{\infty} e^{-x^2} dx$ is convergent.

Student note

6. Show that the integral $\int_1^{\infty} \frac{1+e^x}{x} dx$ is divergent.

Student note

7. Find $\int_0^{\pi} \sec x \, dx$, $\int_{-1}^0 \frac{1}{x^2} \, dx$

Student note

8. Find $\int_1^{\infty} \frac{1}{(3x+1)^2} \, dx$, $\int_1^{\infty} \frac{\ln x}{x^3} \, dx$

Student note

9. Find $\int_0^3 \frac{1}{x\sqrt{x}} dx, \int_0^1 \frac{1}{4y-1} dy$

8.3. Improper integrals of mixed type

- We may be interested in integrating a function to infinity while the function is not continuous on the interval of integration. We call this the improper integrals of mixed type.
- We first split the interval so that f is continuous on that interval and then integrate to infinity

Student note

10. Find $\int_0^{\infty} \frac{1}{4y-1} dy$