2301107 Calculus I

8. Improper Integrals

Outline

Integral formulae

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

Improper integrals

- Sometime a function may not be continuous on $[a, b]$, or a and/or b are not finite. We also interested in

$$\int_a^b f(x)\,dx$$

for example, $\displaystyle\int_0^1 \frac{1}{x}\,dx,\ \int_{-\infty}^0 x e^x\,dx,\ \int_0^3 \frac{1}{x-1}\,dx$

8.1. Improper integrals of Type 1

- <u>Definition</u> of an **improper integral of Type 1**
  - If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then
    $$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$
  - If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then
    $$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

Improper integrals of Type 1

- The improper integrals are convergent if the corresponding limit exists and divergent if the limit does not exist.
- If both $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{a} f(x)\,dx$ are convergent then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Student note

1. Determine whether the integral is convergent or divergent. $\int_1^{\infty}\frac{1}{x}dx$, $\int_1^{\infty}\frac{1}{x^2}dx$, $\int_{-\infty}^{0} x e^x\,dx$

Student note

2. Determine whether the integral is convergent or divergent. $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$, $\int_1^{\infty}\frac{1}{x^p}dx$

8.2. Improper integrals of Type 2

- If f is continuous on $[a, b)$ and is discontinuous at b,
$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$
- If f is continuous on $(a, b]$ and is discontinuous at a,
$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$
- If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ are convergent, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Student note

3. Determine whether the integral is convergent or divergent. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$, $\int_0^{\frac{\pi}{2}} \sec x\,dx$

Student note

4. Determine whether the integral is convergent or divergent. $\int_0^3 \frac{1}{x-1}\,dx$, $\int_0^1 \ln x\,dx$

A comparison test for improper integrals

- Suppose that f and g are continuous functions with $f(x) > g(x) > 0$ for $x \geq a$,
  - If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent
  - If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Student note

5. Show that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

Student note

6. Show that the integral $\int_1^{\infty} \frac{1+e^x}{x}\,dx$ is divergent.

Student note

7. Find $\int_0^{\pi} sec\,x\,dx$, $\int_{-1}^{0} \frac{1}{x^2}dx$

Student note

8. Find $\int_1^{\infty} \frac{1}{(3x+1)^2}dx$, $\int_1^{\infty} \frac{\ln x}{x^3}dx$

Student note

9. Find $\int_0^3 \frac{1}{x\sqrt{x}}dx$, $\int_0^1 \frac{1}{4y-1}dy$

8.3. Improper integrals of mixed type

- We may be interested in integrating a function to infinity while the function is not continuous on the interval of integration. We call this the improper integrals of mixed type.
- We first split the interval so that f is continuous on that interval and then integrate to infinity

Student note

10. Find $\int_0^\infty \frac{1}{4y-1}dy$