

2301107 Calculus I

9. Further Applications of Integration

Chapter 9: Further applications of integration

C09-2

Outline

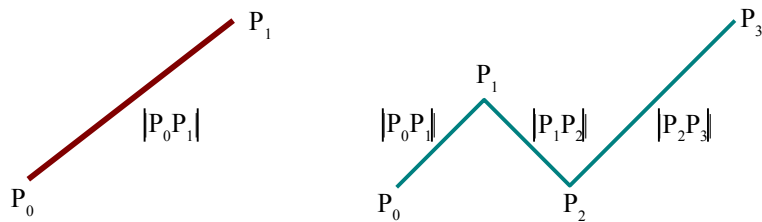
9.1. Arc length of 2-D curves

9.2. Arc length function

9.3. Area of a surface of revolution

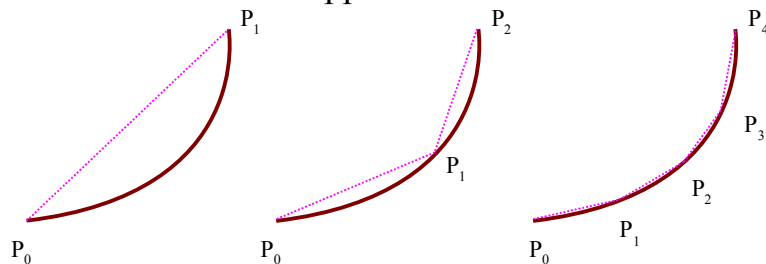
9.1. Length of the polygon curve

- For a straight line, the definition of length is just the euclidean distance between two end points.
- For the polygon curve, the length is the sum of the length of its component lines.



Length of the curve

- Given a curve C defined by $y = f(x)$ where f is continuous and $a \leq x \leq b$.
 - Divide $[a, b]$ into n subintervals with endpoints $x_0, x_1, x_2, \dots, x_n$ and equal width Δx . The point $P_i = (x_i, f(x_i))$ can be obtained to approximate the curve C .



Derivation of the length

- We define the length L of the curve C by

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1} P_i|$$

- If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1} P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

- By the Mean Value Theorem of f ,

$$\begin{aligned} f(x_i) - f(x_{i-1}) &= f'(x_i^*)(x_i - x_{i-1}) \\ \Delta y_i &= f'(x_i^*) \Delta x \end{aligned}$$

Definition of length

- Hence, $|P_{i-1} P_i| = \sqrt{1 + (f'(x_i^*))^2} \Delta x$

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1} P_i| = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + (f'(x_i^*))^2} \Delta x$$

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

- The Arc Length formula: If f is continuous on $[a, b]$ the length of $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

Student note

1. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

Student note

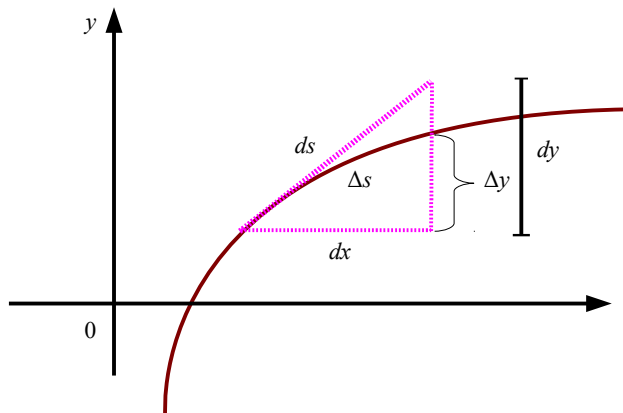
2. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

The arc length function

- Given a smooth curve C having the equation $y = f(x)$, $a \leq x \leq b$.
- Let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.
- We call s the arc length function,

$$s(x) = \int_a^x \sqrt{1 + f'(t)^2} dt$$

Arc length function



Differential of arc length function

- The differential of arc length is

$$ds = \frac{ds}{dx} dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Student note

3. Find the arc length function for the curve $y = 1 + \ln x$ taking $P_0(1, 1)$ as the starting point.

Student note

4. A hawk flying at 15 m/s at an altitude of 180 m. accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

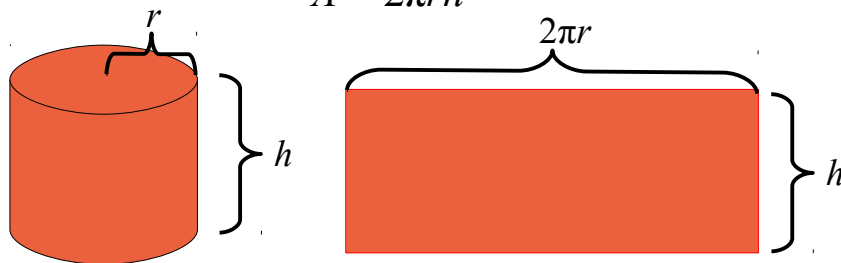
$$y = 180 - \frac{x^2}{45}$$

until it hits the ground where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground.

9.2. Area of a surface of revolution

- A surface of revolution is formed when a curve is rotated about a line. $\int \sqrt{a^2 - x^2} dx$
- Consider the surface of circular cylinder with radius r and height h ,

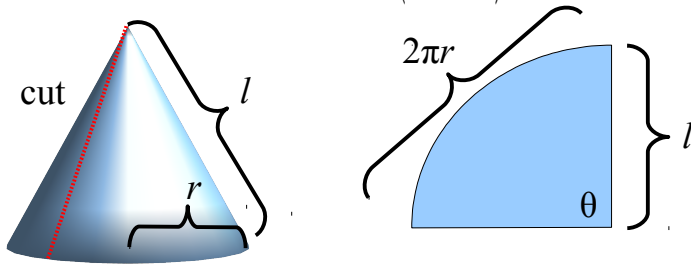
$$A = 2\pi rh$$



Area of a surface of a cone

- A circular cone with a base radius r and slant height l , has the area

$$A = \frac{1}{2} l^2 \theta = \frac{1}{2} l^2 \left(\frac{2\pi r}{l} \right) = \pi r l$$

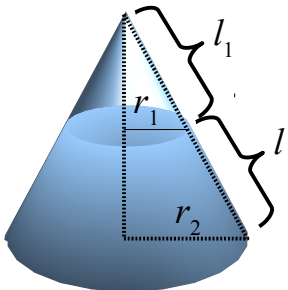


Area of a surface using shell

- We approximate a surface using bands, each formed by rotating a line segment about an axis.

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1)l_1 + r_2 l]$$

- From similar triangles, we have



$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

$$r_2 l_1 = r_1 l_1 + r_1 l \equiv (r_2 - r_1) l_1 = r_1 l$$

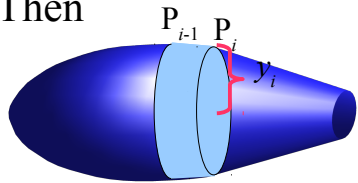
$$A = 2\pi r l$$

Band approximation

- Consider the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x -axis.
- We subdivide $[a, b]$ into n subintervals. The approximate surface is the sum of area of each band.

$$2\pi(y_{i-1} - y_i)/2 |P_{i-1}P_i|$$

- Then



$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx$$

Surface area

- We define the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$ about the x -axis as

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

- If a curve is described as $x = g(y)$, $c \leq y \leq d$, then

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Surface area

- We summarize the notation from

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

as

$$S = \int 2\pi y ds$$

- Moreover,

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

as

$$S = \int 2\pi x ds$$

Student note

5. Find the half portion of the surface of a sphere of radius 2.

Student note

6. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y -axis. Find the area of the resulting surface.

Student note

7. Find the surface area of the ellipsoid which is constructed from rotating the ellipse about the x -axis.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Student note

8. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 80$ ft is given by

$$y = 150 - \frac{(x - 50)^2}{40}$$

Find the distance traveled by the kite.

Student note

9. If the infinite curve $y = e^{-x}$, $x \geq 0$, is rotated about the x -axis, find the area of the resulting surface.