2301107 Calculus I
9. Further Applications of Integration

Outline

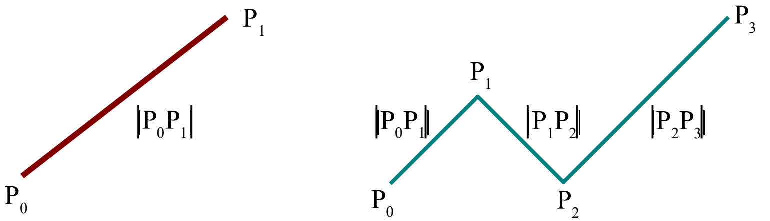
9.1. Arc length of 2-D curves

9.2. Arc length function

9.3. Area of a surface of revolution

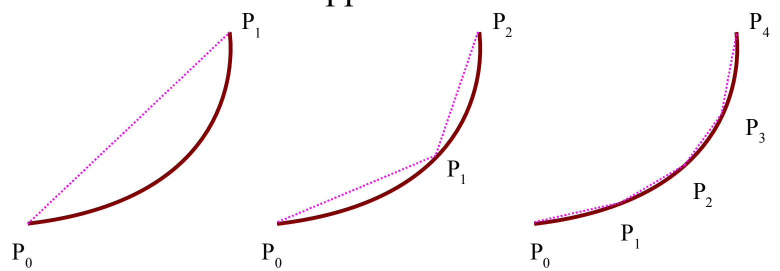
9.1. Length of the polygon curve

- For a straight line, the definition of length is just the euclidean distance between two end points.
- For the polygon curve, the length is the sum of the length of its component lines.



Length of the curve

- Given a curve C defined by $y = f(x)$ where f is continuous and $a \le x \le b$.
  - Divide $[a, b]$ into n subintervals with endpoints $x_0, x_1, x_2, \ldots, x_n$ and equal width Δx. The point $P_i = (x_i, f(x_i))$ can be obtained to approximate the curve C.



Derivation of the length

- We define the length L of the curve C by

$$L=\lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i|$$

- If we let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i|=\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

- By the Mean Value Theorem of f,

$$f(x_i)-f(x_{i-1})=f'(x_i^*)(x_i-x_{i-1})$$
$$\Delta y_i=f'(x_i^*)\Delta x$$

Definition of length

- Hence, $|P_{i-1}P_i|=\sqrt{1+\left(f(x_i^*)\right)^2}\,\Delta x$

$$L=\lim_{n\to\infty}\sum_{i=1}^{n}|P_{i-1}P_i|=\lim_{n\to\infty}\sqrt{1+\left(f(x_i^*)\right)^2}\,\Delta x$$
$$L=\int_a^b\sqrt{1+f'(x_i^*)^2}\,\Delta x$$

- The Arc Length formula: If f is continuous on $[a, b]$ the length of $y = f(x)$, $a \le x \le b$ is

$$L=\int_a^b\sqrt{1+f'(x)^2}\,dx$$

Student note

1. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8).

Student note

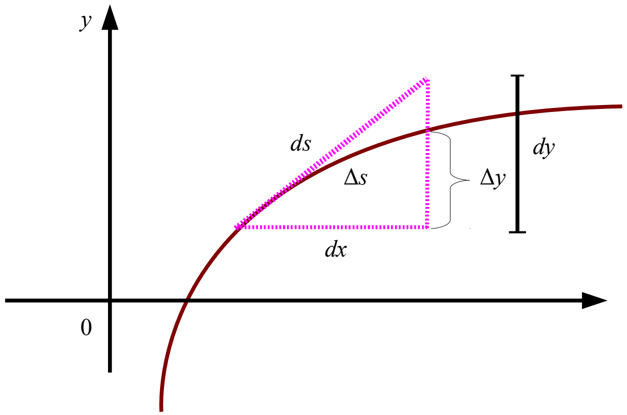
2. Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

The arc length function

- Given a smooth curve C having the equation $y = f(x)$, $a \le x \le b$.
- Let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.
- We call s the arc length function,

$$s(x)=\int_a^x \sqrt{1+f'(t)^2}\,dt$$

Arc length function



Differential of arc length function

- The differential of arc length is

$$ds=\frac{ds}{dx}dx=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$(ds)^2=(dx)^2+(dy)^2$$

$$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Student note

3. Find the arc length function for the curve $y = 1+\ln x$ taking $P_0(1, 1)$ as the starting point.

Student note

4. A hawk flying at 15 m/s at an altitude of 180 m. accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

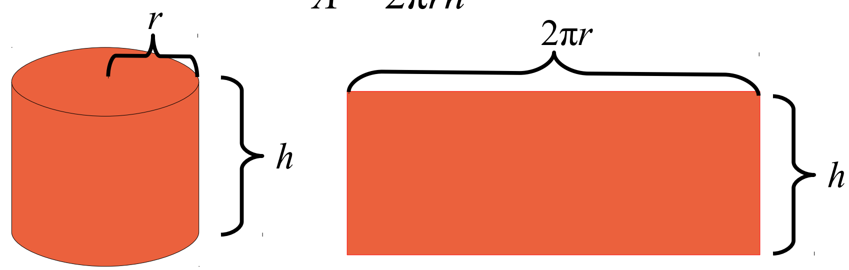
$$y = 180 - \frac{x^2}{45}$$

until it hits the ground where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is droped until the time it hits the ground.

9.2. Area of a surface of revolution

- A surface of revolution is formed when a curve is rotated about a line. $\int \sqrt{a^2 - x^2}\, dx$
- Consider the surface of circular cylinder with radius r and height h,

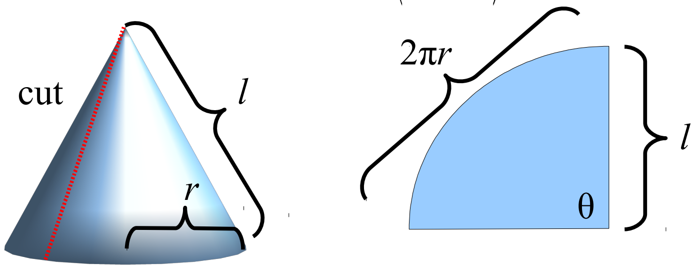
$$A = 2\pi rh$$



Area of a surface of a cone

- A circular cone with a base radius r and slant height l, has the area

$$A = \frac{1}{2} l^2 \theta = \frac{1}{2} l^2 \left(\frac{2\pi r}{l} \right) = \pi r l$$

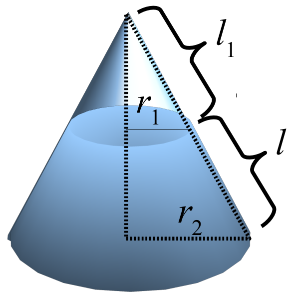


Area of a surface using shell

- We approximate a surface using bands, each formed by rotating a line segment about an axis.

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi[(r_2 - r_1) l_1 + r_2 l]$$

- From similar triangles, we have



$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

$$r_2 l_1 = r_1 l_1 + r_1 l \equiv (r_2 - r_1) l_1 = r_1 l$$

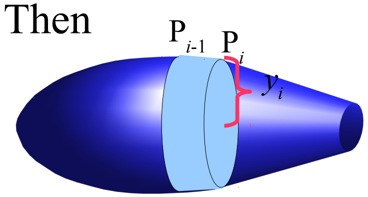
$$A = 2\pi r l$$

Band approximation

- Consider the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis.
- We subdivide $[a, b]$ into n subintervals. The approximate surface is the sum of area of each band.

$$2\pi(y_{i\text{-}1} - y_{\mathrm{i}})/2\,|\,P_{i\text{-}1}P_{\mathrm{i}}|$$

- Then



$$S = \int_a^b 2\pi\, f(x) \sqrt{1 + f'(x)^2}\, dx$$

Surface area

- We define the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis as

$$S = \int_a^b 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- If a curve is described as $x = g(y)$, $c \le y \le d$, then

$$S = \int_c^d 2\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Surface area

- We summarize the notation from

$$S = \int_a^b 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

as

$$S = \int 2\pi y\, ds$$

- Moreover,

$$S = \int_c^d 2\pi\, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

as

$$S = \int 2\pi x\, ds$$

Student note

5. Find the half portion of the surface of a sphere of radius 2.

Student note

6. The arc of the parabola $y = x^2$ from (1, 1) to (2, 4) is rotated about the y-axis. Find the area of the resulting surface.

Student note

7. Find the surface area of the ellipsoid which construct from rotating the ellipse about the x-axis.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$$

Student note

8. A steady wind blows a kite due west. The kite's height above ground from horizontal position x = 0 to x = 80 ft is given by

$$y=150-\frac{(x-50)^2}{40}$$

Find the distance traveled by the kite.

Student note

9. If the infinite curve $y = e^{-x}$, $x \geq 0$, is rotated about the x-axis, find the area of the resulting surface.