Chapter 4

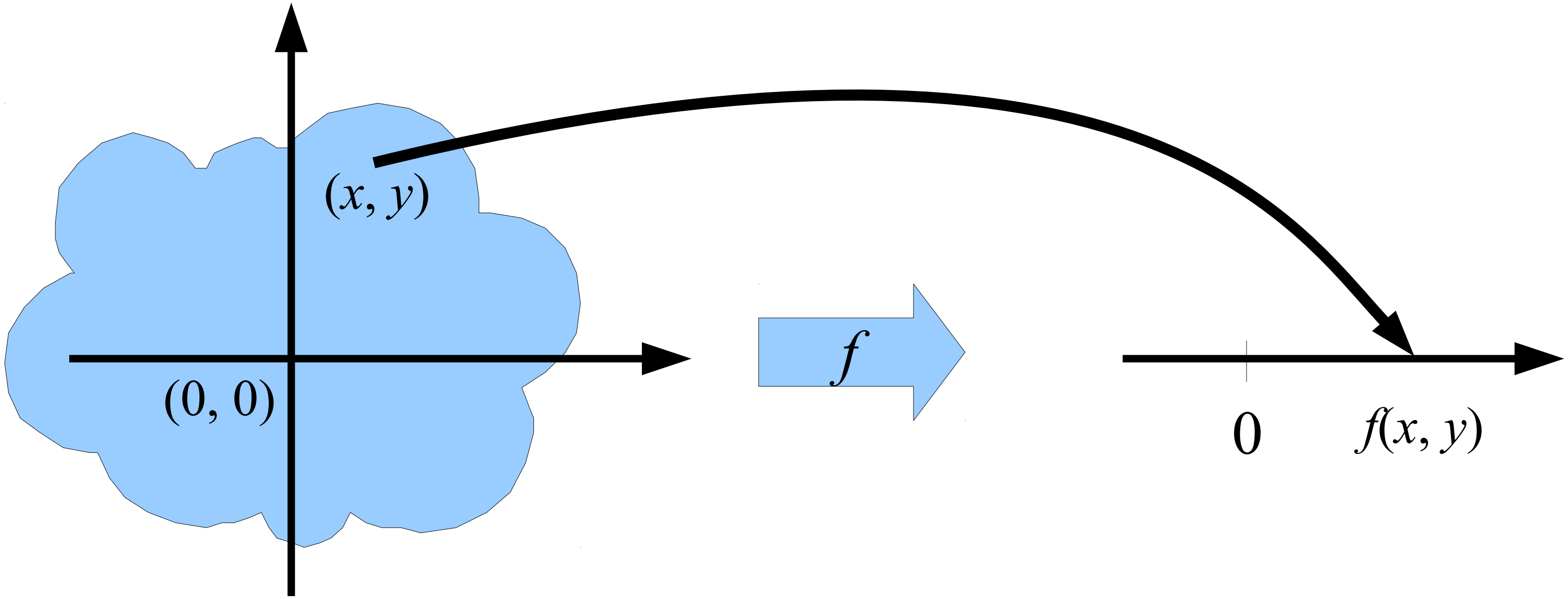
4.1 Functions of several variables

Function of two variables

- A function of two variables is a rule that assigns (x, y) in a set D to a unique real number denoted by $f(x, y)$.
  - D is called the domain of f.
  - Its range is $\{ f(x,y) | (x,y) \in D \}$.

- We often write $z = f(x, y)$ to represent a function
  - x, y are independent variables.
  - z is the dependent variable.
- For a given formula, f, its domain is understood to be a set of all pairs (x, y) that f is defined.

Visualization of $f(x, y)$



Student note

1. Find the domains of the following functions

$(a)\ f(x,y)=y\ln(x-y)$

$(b)\ h(x,y)=\sqrt{\left(\frac{x-1}{x}\right)^2+\left(\frac{y-1}{y}\right)}$

Student note

2. Find the domains of the following functions

$(a)\ f(x,y)=\sqrt{\dfrac{x-y}{x+y}}$

$(b)\ g(x,y)=\left|\dfrac{xy}{x-1}\right|$

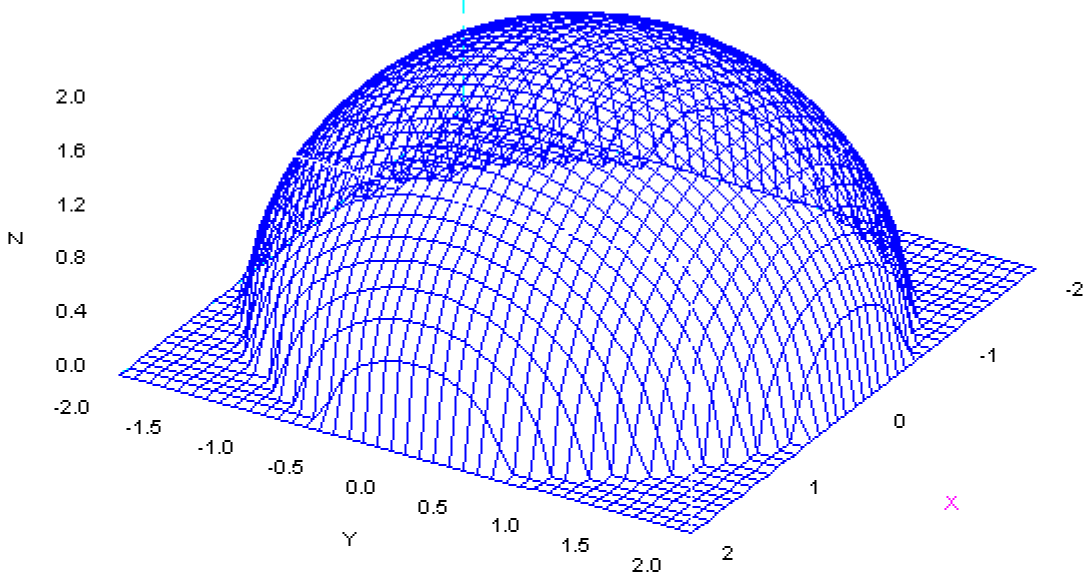
Student note

3. Find the domain and range of $g(x,y)=\sqrt{16-x^2-y^2}$.

Graph

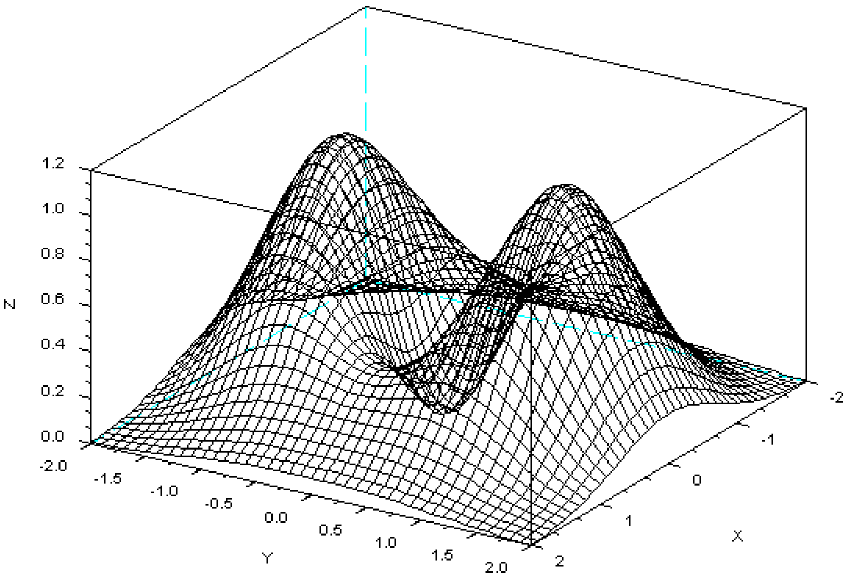
- If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in three-dimensional space such that $z = f(x, y)$.

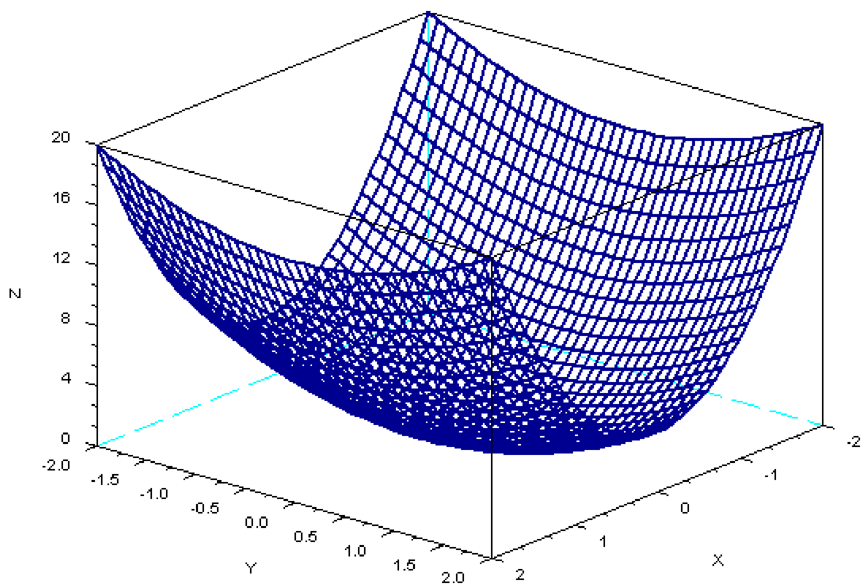
1. Sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.



Sketch the graph of

$$f(x, y) = (x^2 + 3y^2)e^{-x^2 - y^2} \qquad g(x, y) = 4x^2 + y^2$$

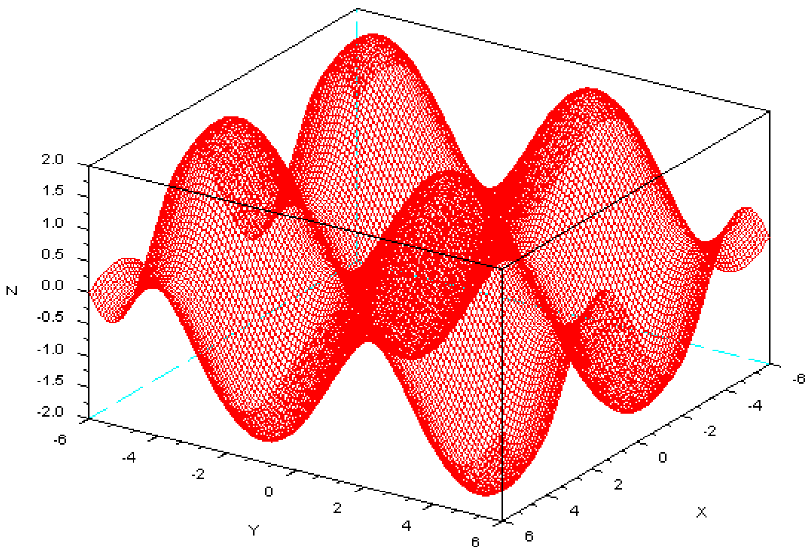


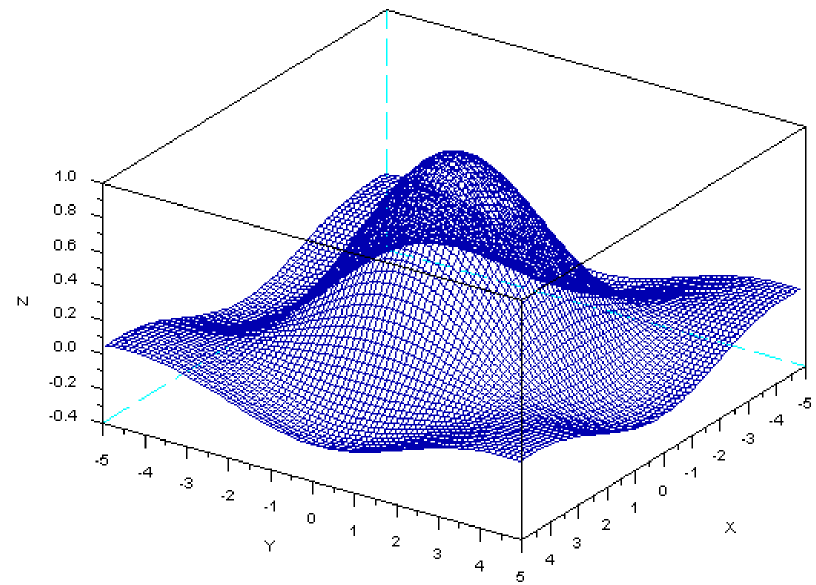


Sketch the graph of

$$f(x, y) = \sin x + \sin y$$

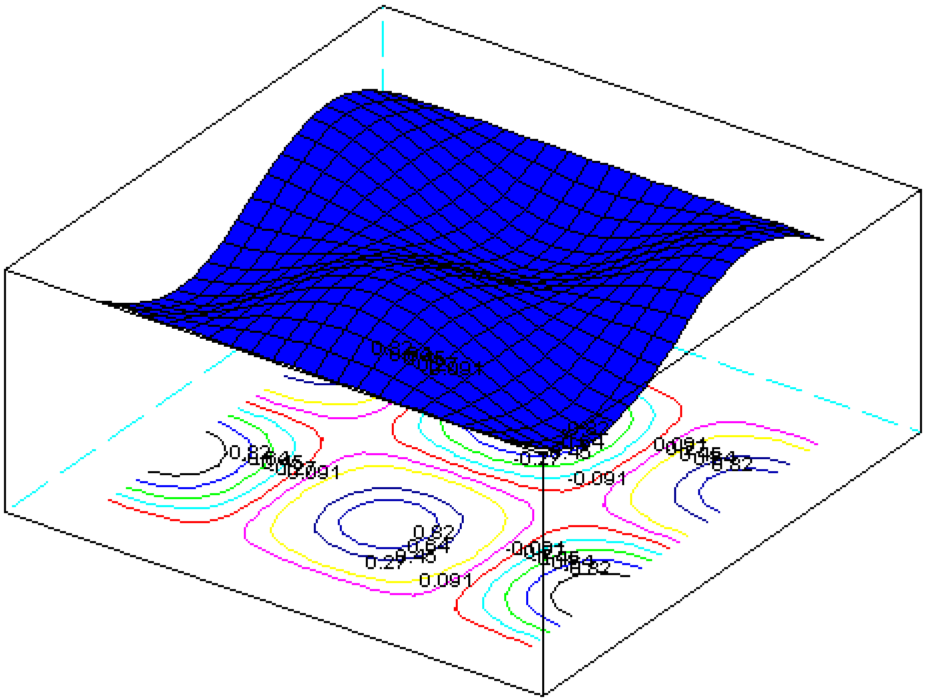
$$g(x, y) = \frac{\sin x \sin y}{x\, y}$$



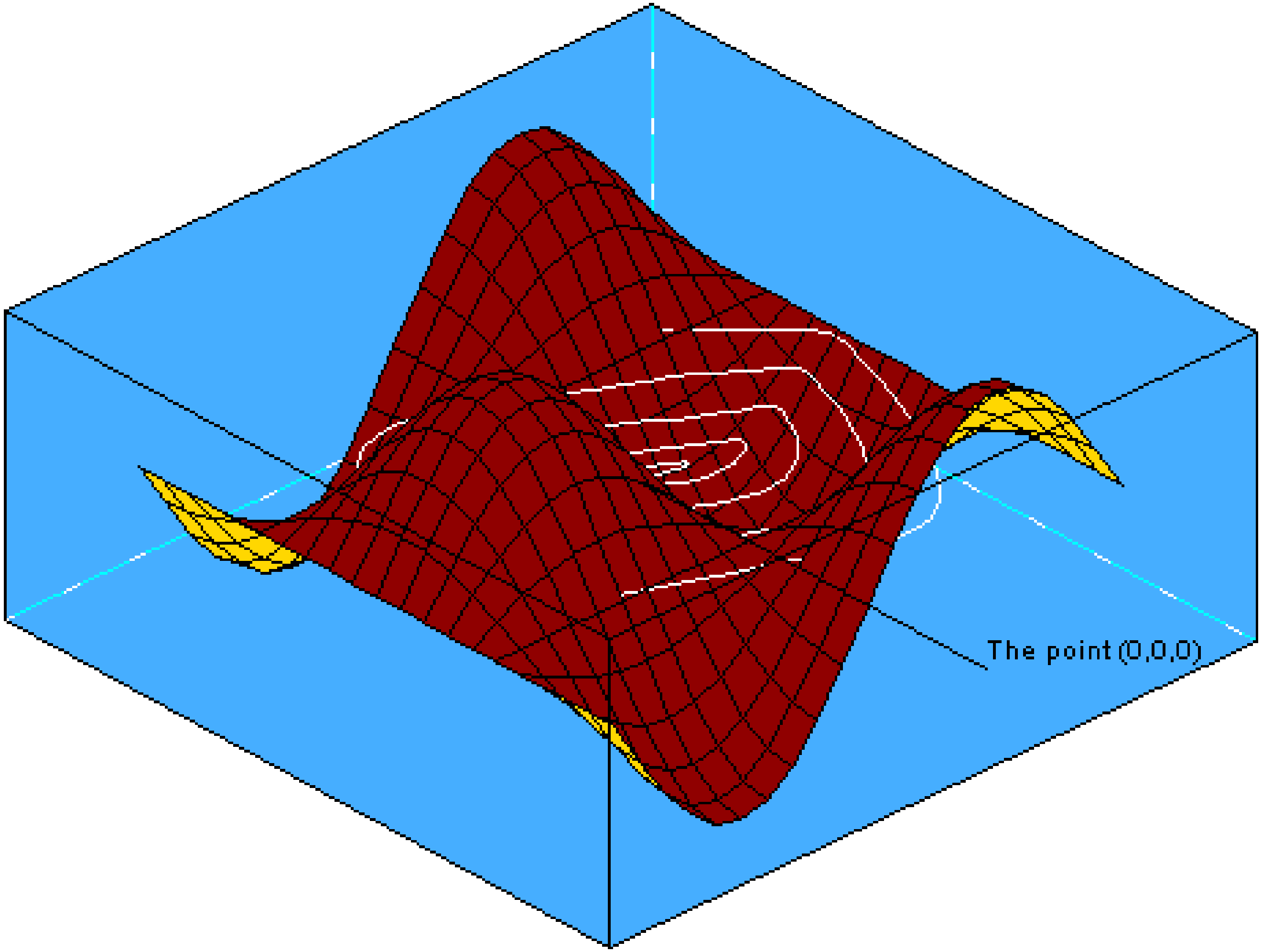


Level curves

- The level curves of a function *f* of two variables are the curves with equation $f(x,y) = k$ where k is a constant. (in the range of *f*)



Level curves



Functions of three or more variables

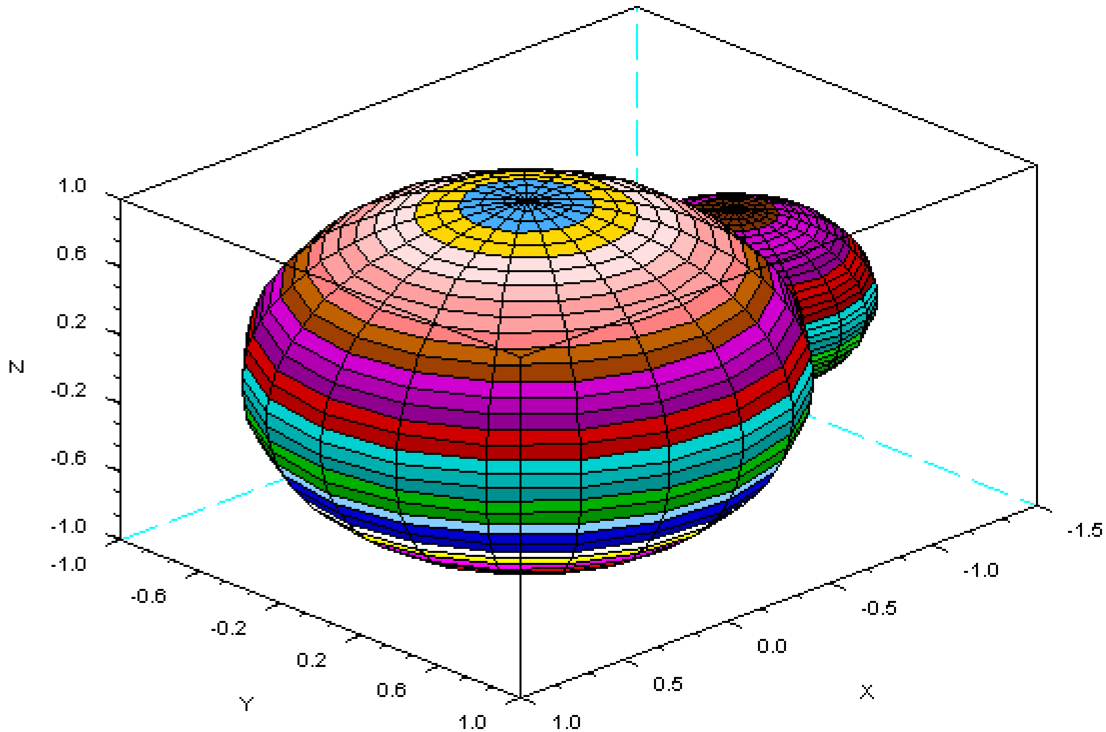
- A function of three variables, f, is a rule that assigns (x, y, z) in a domain, D, to a unique real number denoted by $f(x, y, z)$.

1. Find the domain of f if $f(x, y, z) = \ln(y+x) + xy \cos y$.

Student note

2. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$



Student note

3. Find the domain and range of $f(x, y, z) = e^{\sqrt{z - x^2 - y^2}}$.

Student note

4. Find and sketch the domain of the functions:

$$f(x,y)=\sqrt{x}+\sqrt{y}$$

$$f(x,y)=\sqrt{y-x}\ln(y+x)$$

Student note

5. Match the function with its graph.

(a) $f(x, y) = |x| + |y|$

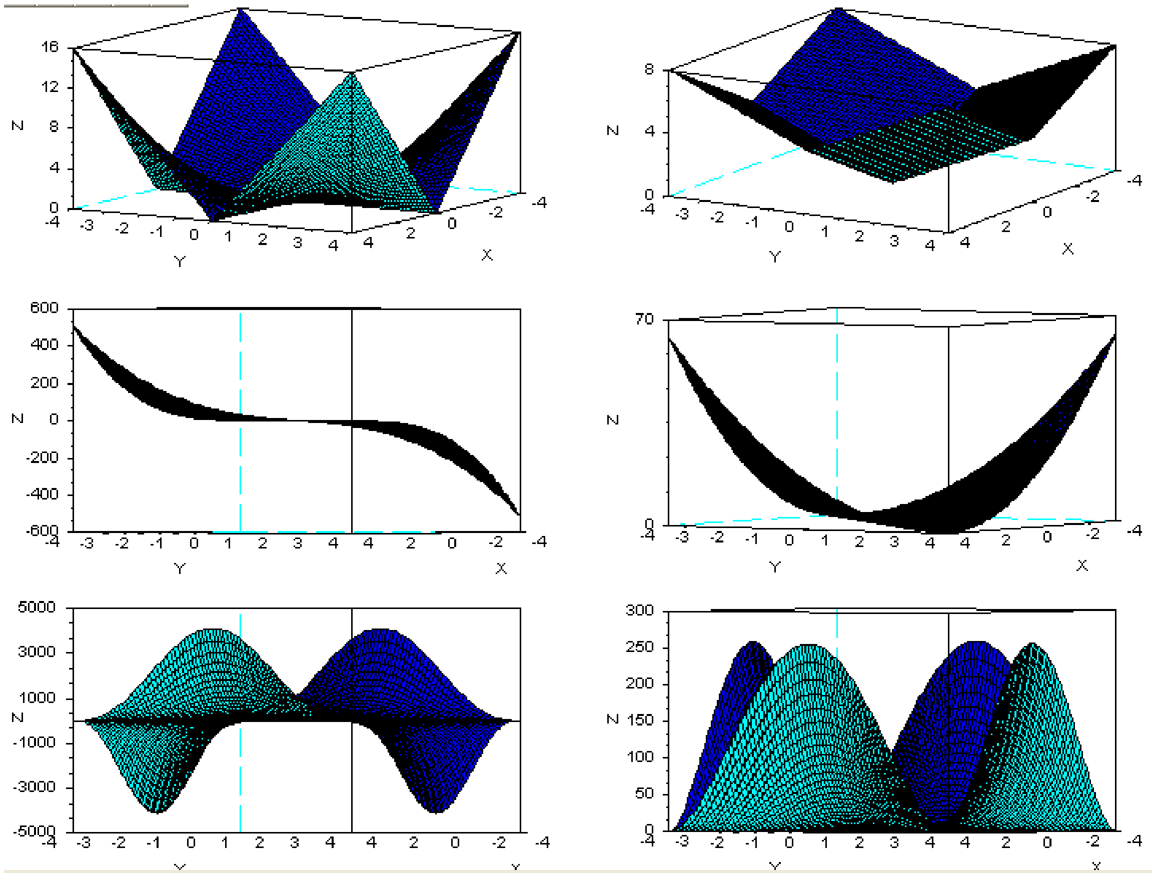
(b) $f(x, y) = (x - y)^2$

(c) $f(x, y) = |x\,y|$

(d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^3$

(f) $f(x, y) = (x^2 - y^2)^3$



Chapter 4

4.2 Limits and Continuity

Limit

- Let f be a function of two variables whose domain D includes point arbitrarily close to (a, b). The limit of $f(x, y)$ as (x, y) approach (a, b) is L is written as

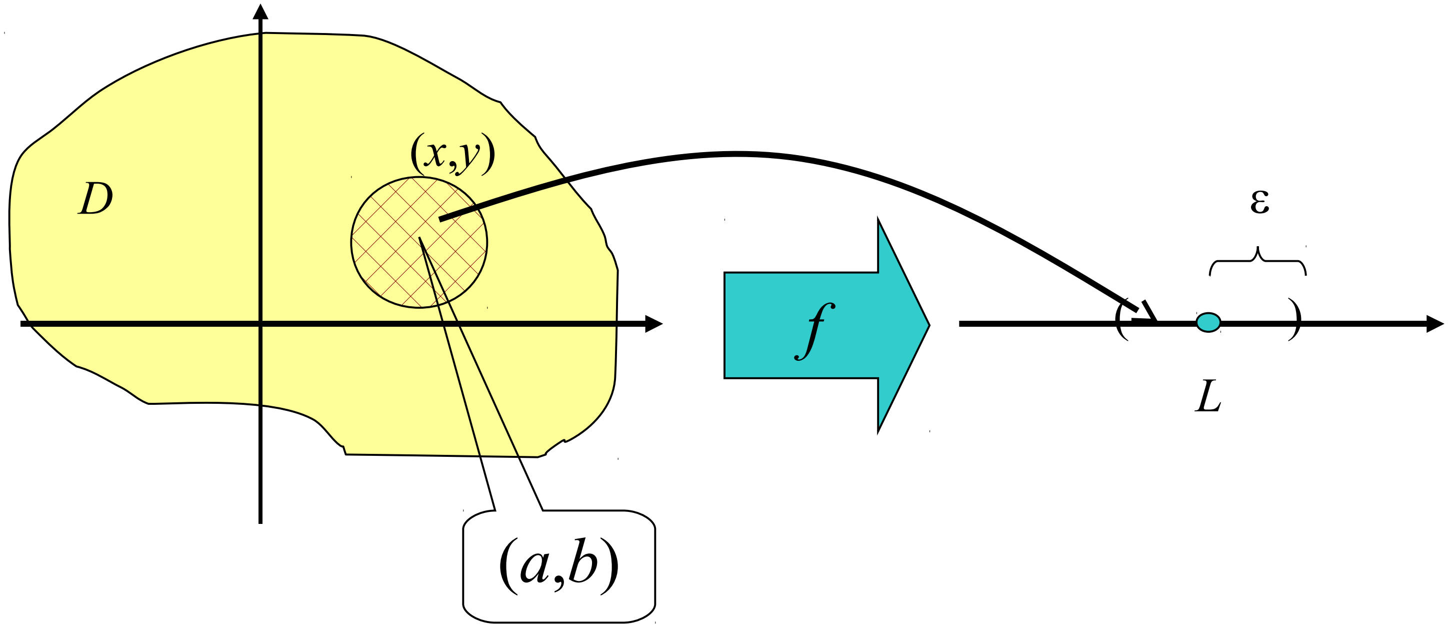
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means for $\varepsilon > 0$, there is a corresponding $\delta > 0$,

$|f(x, y) - L| < \varepsilon$ whenever (x, y) in D and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

Limit concept



Non-existence of limit at a point

- If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to \mathrm{L}_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$ then the limit does not exists.

1. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exists.

Student note

2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exists.

Student note

3. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exists.

Theorem

- Theorem: Given a function *f*, *g* of two variables, if there exists M and δ such that

$$|f(x,y)| \le M, \text{ for } (x,y) \in D_f \cap B'((x_0,y_0),\delta)$$

and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = 0$

- Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = 0$$

where $B'((x_0,y_0),\delta) = \{(x,y) \mid 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$

Student note

1. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2} = 0.$

Theorem

- Given functions f and g of two variables,
  - If $f(x, y) = c$ and c is a constant then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = c$$

  - If $f(x, y) = x$ then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0$$

  - If $f(x, y) = y$ then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = y_0$$

Limit properties

- **Theorem**: If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y)=B$

$$(1)\ \lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)+g(x,y)\right)=A+B$$

$$(2)\ \lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)-g(x,y)\right)=A-B$$

$$(3)\ \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)}=\frac{A}{B}\ if\ B\neq 0$$

$$(4)\ \lim_{(x,y)\to(x_0,y_0)} |f(x,y)|=|A|$$

$$(5)\ \lim_{(x,y)\to(x_0,y_0)} \sqrt{f(x,y)}=\sqrt{A}\ when\ A\geq 0$$

Student note

1. Determine $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y^2}{x^2+y^2}$.

Student note

2. Determine $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$.

Student note

3. Determine $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{\sqrt{x^2+y^2}}$.

Definition

- A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- We say f is continuous on D if f is continuous at every point (x, y) in D.

Student note

1. Determine the continuity of $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$.

Student note

2. Determine the continuity of $\arctan\left(\dfrac{y}{x}\right)$.

Student note

3. Determine the continuity of $f(x, y)$ when

$$f(x,y)=\begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 0 & \text{if} \quad (x,y)=(0,0) \end{cases}$$

Continuity of functions

- The continuity of the functions of more than two variables can be extended easily. For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x_1, x_2, \ldots, x_n) - L| < \varepsilon$$

whenever $0 < \|(x_1, x_2, \ldots, x_n) - (a_1, a_2, \ldots, a_n)\| < \delta$.

- where $(a_1, a_2, \ldots, a_n)$ is the point to determine the limit.

$$\lim_{(x_1, x_2, \ldots, x_n) \to (a_1, a_2, \ldots, a_n)} f(x_1, x_2, \ldots, x_n) = L$$

- f is continuous at $(a_1, a_2, \ldots, a_n)$ if

$$\lim_{(x_1, x_2, \ldots, x_n) \to (a_1, a_2, \ldots, a_n)} f(x_1, x_2, \ldots, x_n) = f(a_1, a_2, \ldots, a_n)$$

Student note

1. Find $h(x,y) = g(f(x,y))$ and the set on which h is continuous.

$(1)\ g(t)=t^3-\sqrt{t},\ f(x,y)=x+y-1$

$(2)\ g(t)=\dfrac{\sqrt{t}+1}{\sqrt{t}-1},\ f(x,y)=y^2-x$

Student note

2. Determine the set of points at which the function is continuous:

$$f(x,y)=\frac{\cos(xy)}{e^x-xy}$$

$$f(x,y)=\frac{x-y}{1+x^2+y^2}$$

Student note

3. Determine the continuity of

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{2x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 1 & \text{if} \quad (x,y)=(0,0)\end{cases}$$

Student note

4. Determine the continuity of

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 1 & \text{if} \quad (x,y)=(0,0) \end{cases}$$

Chapter 4

4.3 Partial derivatives

Definition

- A function $f(x, y)$ can be considered as a function with one variable x if y is fixed, say $y = b$ then $g(x) = f(x, b)$.
- The derivative of g at a is called the partial derivative of f with respect to x at (a, b) denoted by $f_x(a, b)$.

$$f_x(a, b) = g'(a) \text{ where } g(x) = f(x, b)$$

- By the definition of derivative,

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Definition

- A function $f(x, y)$ can be considered as a function with one variable y if x is fixed, say $x = a$ then $h(y) = f(a, y)$.
- The derivative of h at b is called the partial derivative of f with respect to y at (a, b) denoted by $f_y(a, b)$.

$$f_y(a, b) = h'(b) \text{ where } h(y) = f(a, y)$$

- By the definition of derivative,

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

Notations for partial derivatives

- If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

- Rule for finding partial derivatives, $z = f(x,y)$
  - To find f_x regard y as a constant and differentiate $f(x,y)$ with respect to x.
  - To find f_y regard x as a constant and differentiate $f(x,y)$ with respect to y.

Student note

1. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ of the following

(1) $z = f(x) + g(y)$

(2) $z = f(x + y)$

(3) $z = f(x)\, g(y)$

(4) $z = f(xy)$

Student note

2. Use the implicit differentiation to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

(1) $x^2 + y^2 + z^2 = 3xyz$

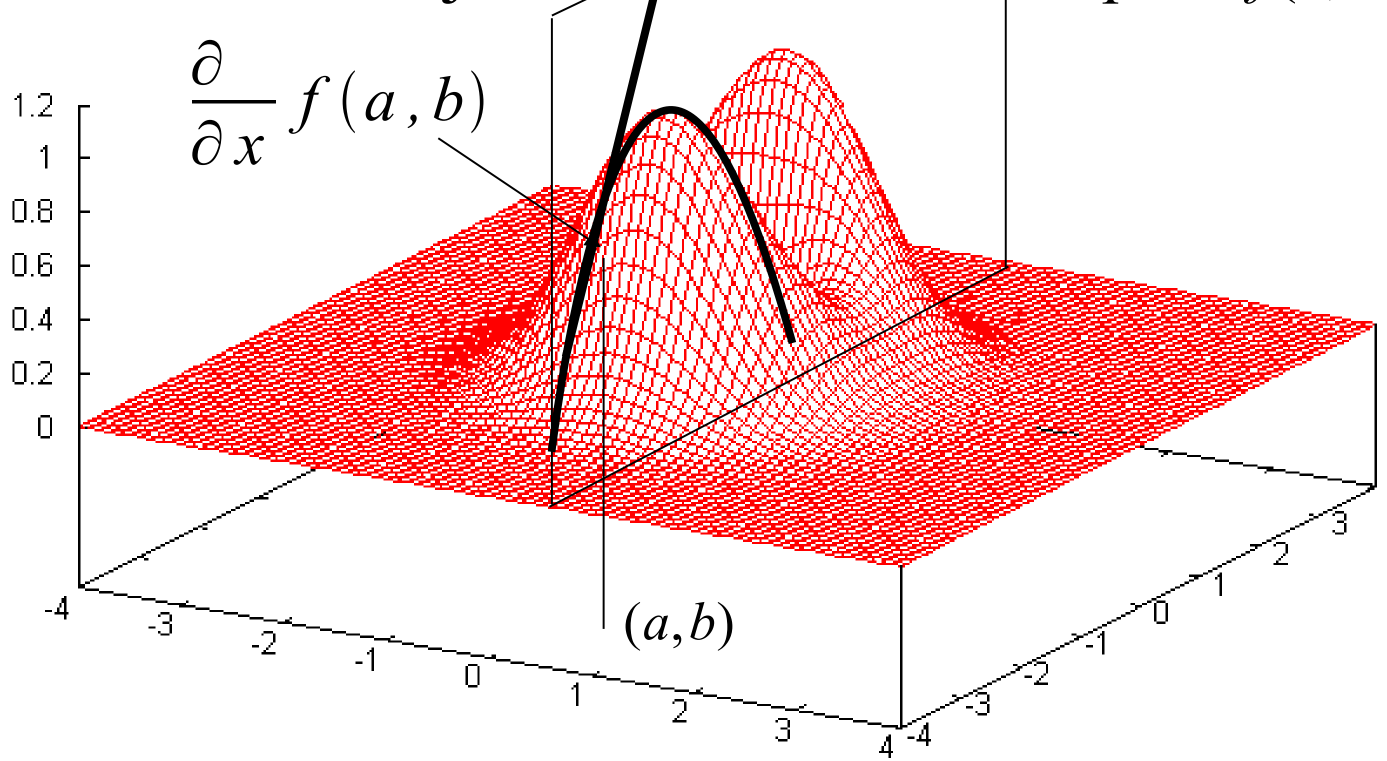
(2) $x - y = \arctan(yz)$

(3) $\sin(x,y,z) = x + 2y + 3z$

(4) $yz = \ln(x + z)$

Interpretations of partial derivatives

- For fix $y = b$ (plane), the partial derivative is just slope of $f(x, b)$



Student note

1. Calculate $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ for $f(x, y) = \cos\left(\dfrac{x}{x+y}\right)$.

Functions of more than two variables

- If $w = f(x,y,z)$, then the partial derivative with respect to x is

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

- In general, if $u = f(x_1, x_2, \ldots, x_n)$, the partial derivative with respect to x_i is

$$f_{x_i}(x_1, x_2, \ldots, x_i, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h}$$

Student note

1. Find f_x, f_y, f_z if $f(x,y,z) = e^{xy} \ln z$.

Higher derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y is also functions of two variables, so there will be the second partial derivatives of f.

$$\left(f_x\right)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$\left(f_x\right)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\left(f_y\right)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$\left(f_y\right)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's theorem

- **Theorem**: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

1. Find the second partial derivatives of

$$f(x, y) = x^2 - xy + y^3$$

Student note

1. Calculate f_{yxzx} for $f(x, y, z) = \cos(x - y + yz)$.

Student note

- Given a function

$$f(x,y)=\begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

- Use the definition to show that
  - $D_1 f(0, y) = -y$
  - $D_2 f(x, 0) = x$
  - $D_{12} f(0, 0) \neq D_{21} f(0, 0)$

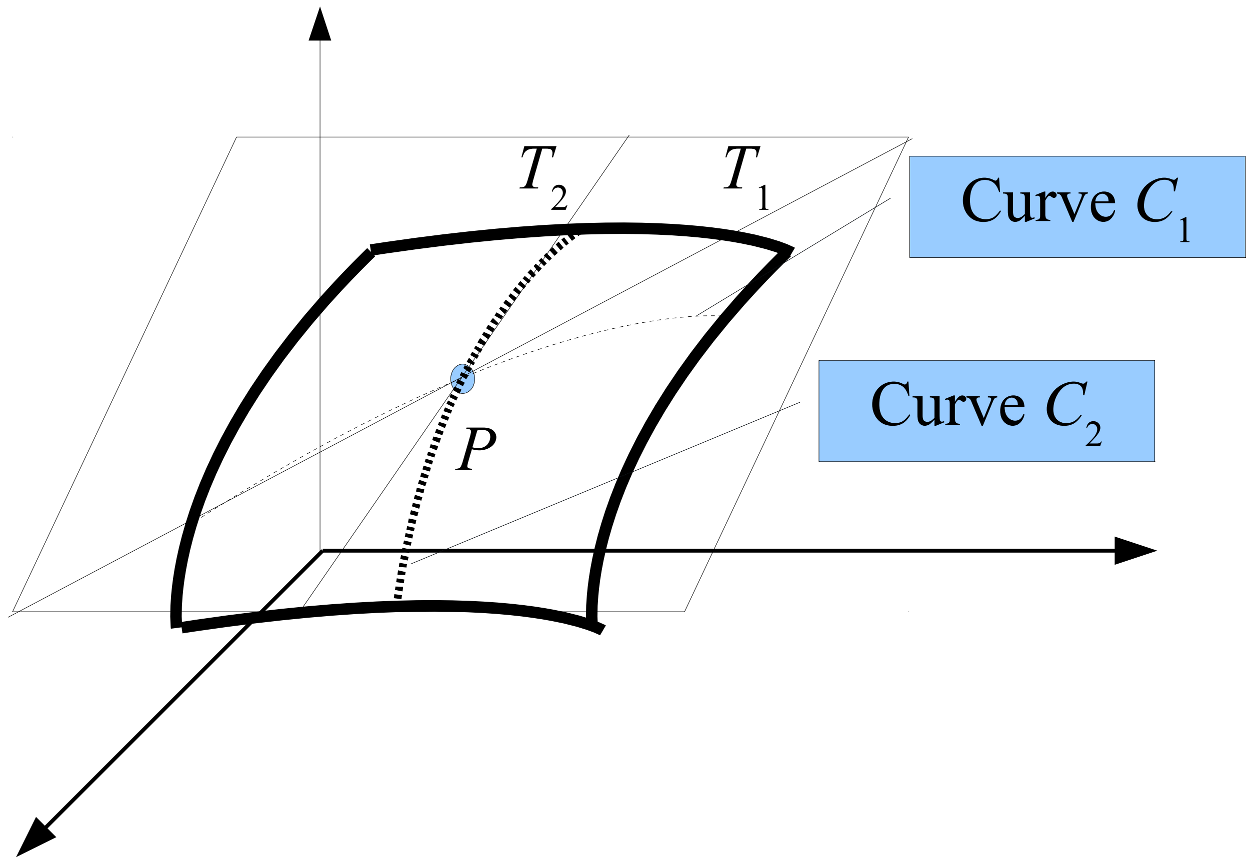
Chapter 4

4.4 Tangent Planes and Linear Approximations

Tangent planes

- Suppose a surface S has equation $z = f(x, y)$ and a point $P(x_0, y_0, z_0)$ is on a surface S.
- Let C_1 and C_2 be the curves formed by the intersection of the vertical planes $y = y_0$ and $x = x_0$ with the surface S.
- Note that the point P lies on both C_1 and C_2.
- Let the tangent lines of C_1 and C_2 are T_1 and T_2, the tangent plane to the surface S at the point P is defined as the plane that contains both tangents.

Tangent plane



Lemma

- Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

1. Find the tangent plane to the elliptic paraboloid at the point (1, -1, 4) where $z = x^2 + 3y^2$.

Linear approximations

- Given $z = f(x,y)$, the linear approximation of $z = f(x,y)$ at (a,b) is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- The linear function whose graph is this tangent plane,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- We call $L(x,y)$ the linearization of f at (a,b).

Definition

- If $z = f(x,y)$ then f is differentiable at (a,b) if Δz can be expressed in the form

$$\Delta z = f_x(a,\ b)\Delta x + f_y(a,\ b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

  where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

- Note that a differentiable function is one for which the linear approximation is a good approximation when $(x,\ y)$ is closed to $(a,\ b)$.

Theorem

- **Theorem**: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

1. Show that $f(x, y) = ye^{xy}$ is differentiable at (1, 0) and find its linearization to approximate $f(1, -0.1)$.

Differentials

- If $z = f(x, y)$, we define the differentials dx and dy to be independent variables, then the total differential dz is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

- The linear approximation can be written as

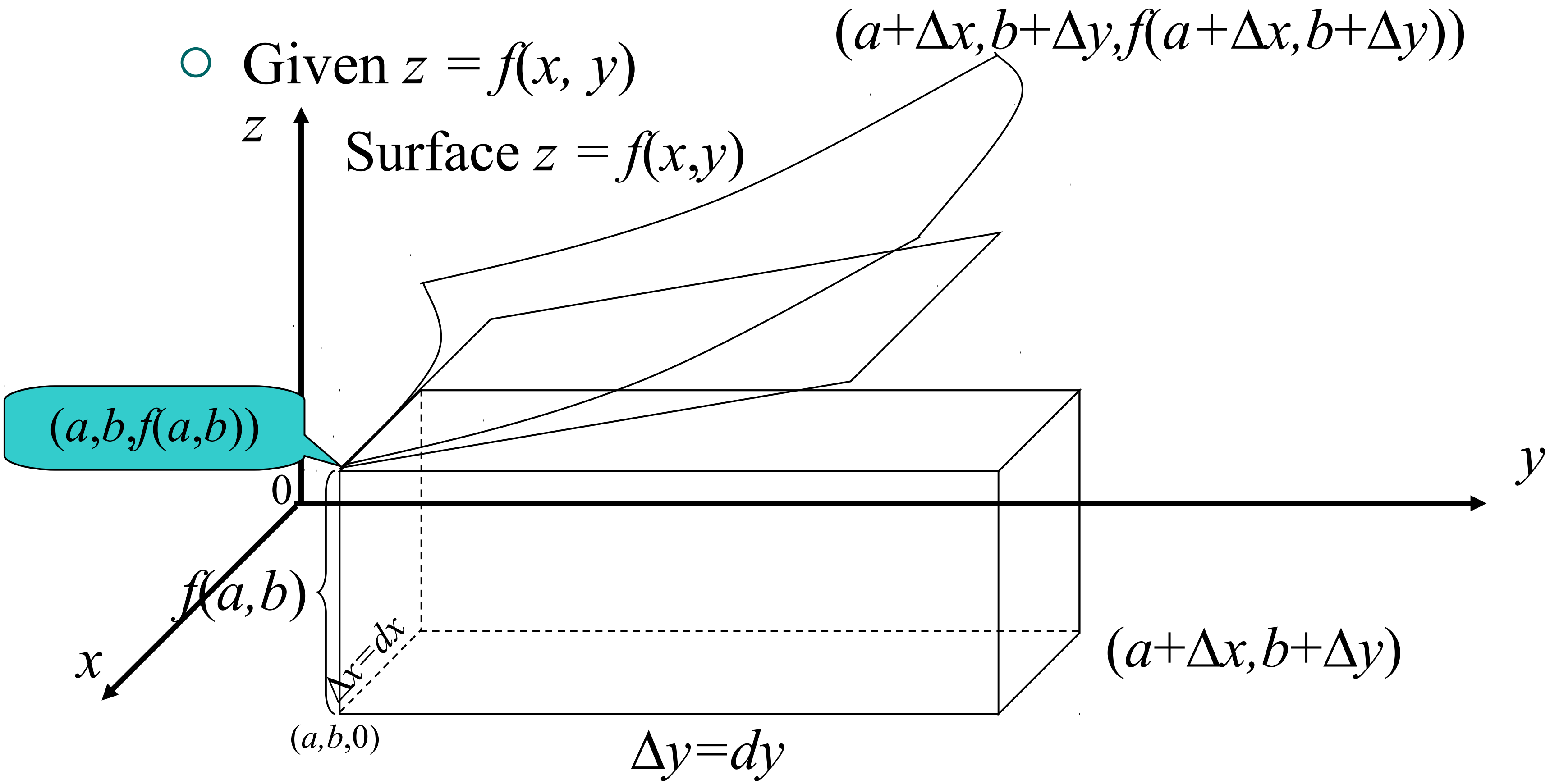
$$\begin{aligned} f(x,y) &\approx f(a,b) + dz \\ &\approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \end{aligned}$$

Student note

1. The base radius and height of a right circular cone are measured as 10 cm. and 25 cm., respectively, with a possible error in measurement of as much as 0.1 cm. in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Differentials

- Given $z = f(x, y)$



$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Functions of three or more variables

- If $w = f(x,y,z)$, then the linear approximation is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

- The differential dw is defined as

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + + \frac{\partial w}{\partial z}dz$$

Student note

1. The dimensions of a rectangular box are measured to be 25cm., 30 cm. and 20 cm., and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Chapter 4

4.5 The Chain rule

The chain rule (1)

- Case 1: Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

1. If $z = 2x^2y + y^2$ where $x = sin(t)$, $y = \cos(3t)$ find dz/dt when $t = 0$.

The chain rule (2)

- Case 2: Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1. If $z = \ln(xy + x^2)$ where $x = st$, $y = s + t$ find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

The chain rule (3)

- General version: Suppose that u is a differentiable function of n variables and each x_i are differentiable function of m variables, then u is a function of m variables,

$$\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\cdots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

1. If $u=xy+zx^2$ where $x=rs\,e^t$, $y=r^2st$, $z=s\cos(t)$ find $\frac{\partial u}{\partial s}$.

Implicit differentiation

- Suppose $F(x, y) = 0$ is given where y is implicitly defined by x and F is differentiable,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

- Assume that $\frac{\partial F}{\partial y}$ is not zero,

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Implicit differentiation

- Suppose $F(x, y, z) = 0$ is given where z is implicitly defined by x, y and F is differentiable,

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial z}\frac{dz}{dx}=0.$$

- Assume that $\frac{\partial F}{\partial z}$ is not zero,

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}},\ \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Student note

1. Determine $\dfrac{dy}{dx}$ of $x^4 + y^4 = 8xy$.

Student note

2. Determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 = 4xz - 4yz + 1.$

Student note

3. Determine the formula for $\dfrac{\partial^2 z}{\partial s \partial t}$ for $z = f(x, y)$ and $x = g(s,t)$ and $y = h(s,t)$.

Student note

4. If $z = f(x - y)$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$

Chapter 4

4.6 Directional Derivatives and the Gradient Vector

Directional derivatives

- Definition: The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\boldsymbol{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_{0,} y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_{0,} y_0)}{h}$$

  if this limit exists.

- **Theorem**: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\boldsymbol{u} = \langle a, b \rangle$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Student note

1. Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\boldsymbol{u}$ is the unit vector given by angle $\theta = \pi/6$.

What is $D_u f(1, 2)$?

The gradient vector

- Definition: If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}.$$

1. If $f(x, y) = \sin x + e^{xy}$, what is $\nabla f(x, y)$?

Directional derivatives

- Definition: The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\boldsymbol{u} = \langle a, b, c \rangle$ is

$$D_{\vec{u}} f(x_{0,} y_{0,} z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_{0,} y_{0,} z_0)}{h}$$

if this limit exists.

- The vector notation for the directional derivative is

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

- We write $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

Maximizing the directional derivative

- **Theorem**: Suppose f is differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(\boldsymbol{x})$ is $|\nabla f(\boldsymbol{x})|$ and it occurs when u has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.

1. If $f(x, y) = xe^y$, find the rate of change of f at the point P(2, 0) in the direction from P to $Q(\frac{1}{2}, 2)$. In what direction does f have the maximum rate of change? What is this maximum rate of change?

Tangent planes to level surfaces

- Suppose S is a surface with (the level surface) equation $F(x, y, z) = k$.
- Let $P(x_0, y_0, z_0)$ be a point on S and let C, $\boldsymbol{r}(t) = <x(t), y(t), z(t)>$, be any curve that lies on the surface S and passes through the point P at t_0, $\mathbf{r}(t_0) = <x_0, y_0, z_0>$.

$$F(x(t), y(t), z(t)) = k$$

- Using the chain rule, we have

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0.$$

- Written as $\nabla F \bullet \boldsymbol{r}'(t) = 0.$

Tangent planes to level surfaces

- When $t = t_0$, we have $\boldsymbol{r}(t_0) = <x_0, y_0, z_0>$, so

$$\nabla F(x_0, y_0, z_0) \bullet \boldsymbol{r}'(t_0) = 0.$$

- The gradient vector at P, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\boldsymbol{r}'(t_0)$ to any curve C on S that passes through P.
- The tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ has the equation

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

- The normal line to S at P is the line passing through P and perpendicular to the tangent plane.

Student note

1. Find the equations of the tangent plane and normal line at the point (-2, 1, -3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Student note

2. Find the directional derivative of f at the given point in the direction indicated by the angle θ.

   2.1 $f(x, y) = x^2y^3 - y^4$, $(2, 1)$, $\theta = \pi/4$.

   2.2 $f(x, y) = x \sin(xy)$, $(2, 0)$, $\theta = \pi/3$.

Student note

3. Find the gradient of f and the rate of change of f at P in the direction of the vector $\boldsymbol{u}$.

   3.1 $f(x, y) = 5xy^2 - 4x^3y$, $P(1, 2)$, $\boldsymbol{u} = \langle 5/13, 12/13 \rangle$.

   3.2 $f(x, y) = y \ln(x)$, $P(1, -3)$, $\boldsymbol{u} = \langle -4/5, 3/5 \rangle$.

Student note

4. Find the maximum rate of change of f at the given point and the direction in which it occurs.

   4.1 $f(p, q) = qe^{-p} + pe^{-q}$, $(2, 4)$.

   4.2 $f(x, y, z) = x^2y^3z^4$, $(1, 1, 1)$.

Chapter 4

4.7 Maximum and Minimum Values

Local maximum & minimum

- **<u>Definition</u>**: A function of two variables has a *local maximum* at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b), "(x, y) in some disk with center (a, b)". The number $f(a, b)$ is called a *local maximum value*.
- **<u>Definition</u>**: A function of two variables has a *local minimum* at (a, b) if $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a *local minimum value*.
- If the inequalities hold for all points (x, y) in the domain of f, then f has an *absolute maximum* (*absolute minimum*) at (a, b).

Theorem

- Theorem: If f has a local maximum or minimum at (a, b) and the first-order derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
- A point (a, b) is called a *critical point* (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of these partial derivatives does not exist.

Student note

1. Find the extreme values of $f(x, y) = y^2 - x^2$.

Second Derivatives Test

- Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$$

  - If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
  - If D > 0 and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
  - If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. In this case, the point (a, b) is called a *saddle point* of f.

Student note

1. Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Student note

2. Find and classify the critical points of the function
   $f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$.

Student note

3. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum value of such a box.

Closed set

- A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.
- A closed set in $\mathbb{R}^2$ is a set that contains all its boundary points.
- For example, the disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$ which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set.
- A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.

Extreme value theorem

- If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.
- To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:
  - Find the values of f at the critical points of f in D.
  - Find the extreme values of f on the boundary of D.
  - The largest of the values from previous steps is the absolute maximum value; the smallest of these values is the absolute minimum value.

Student note

1. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Student note

2. Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = x^3y + 12x^2 - 8y$.

Student note

3. Find the absolute maximum and minimum values on the set $D = \{(x, y) \mid |x| < 1, |y| < 1\}$ where $f(x, y) = x^2 + y^2 + x^2y + 4$.

Student note

4. Find the points on the surface $x^2y^2z = 1$ that are closest to the origin.

Student note

5. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Student note

6. Find three positive numbers, x, y, z whose sum is 100 such that $x^a y^b z^c$ is a maximum.

Student note

7. Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

Student note

8. Find an equation of the plane that passes through the point (1, 2, 3) and cuts off the smallest volume in the first octant.