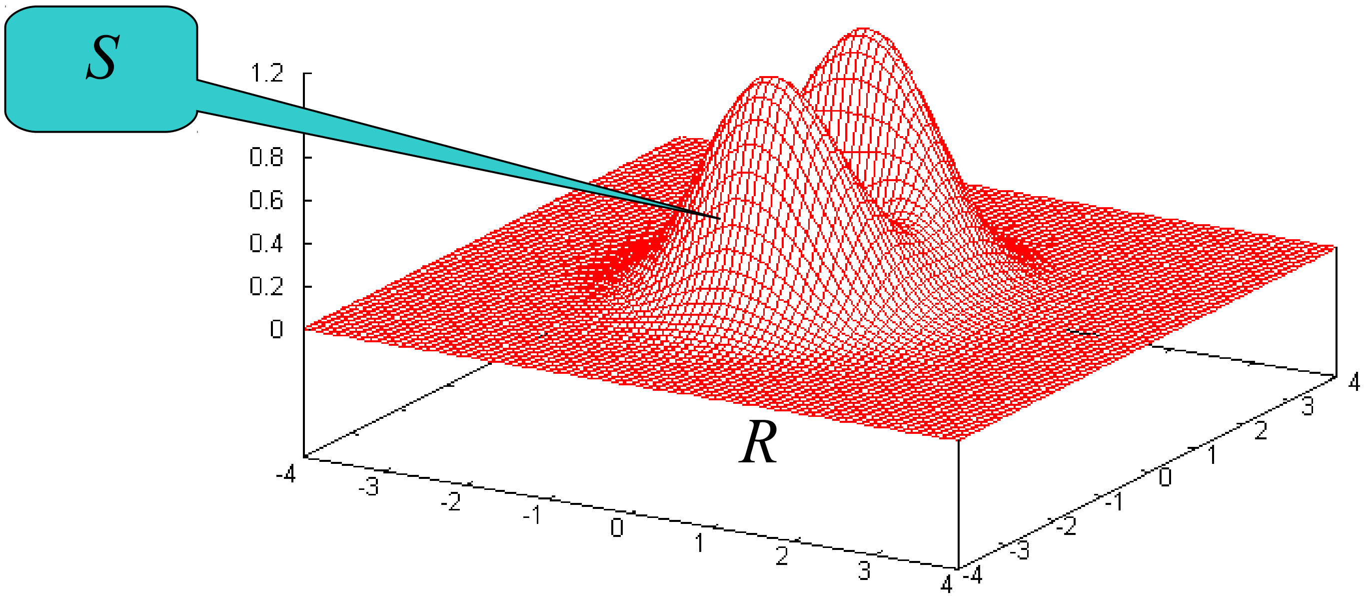
Chapter 5

5.1 Double integrals over rectangles

Volumes

- Consider a nonnegative function f defined on $R = [a, b] \times [c, d] = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$
- Let S be the solid that lies above xy plane

$$S = \{(x, y, z) \mid 0 \leq z \leq f(x,y), (x,y) \in R\}.$$



Volume over rectangles

- Divide R into subrectangles.

  $R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i, y_{i-1} \le y \le y_i\}$
- Each with area $\Delta A = \Delta x \times \Delta y$
- Choose a sample point (x_{ij}^*,y_{ij}^*) in each R_{ij}, we can approximate the volume by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

- As m, n are getting larger the volume becomes

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Definition

- The double integral of *f* over the rectangle *R* is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

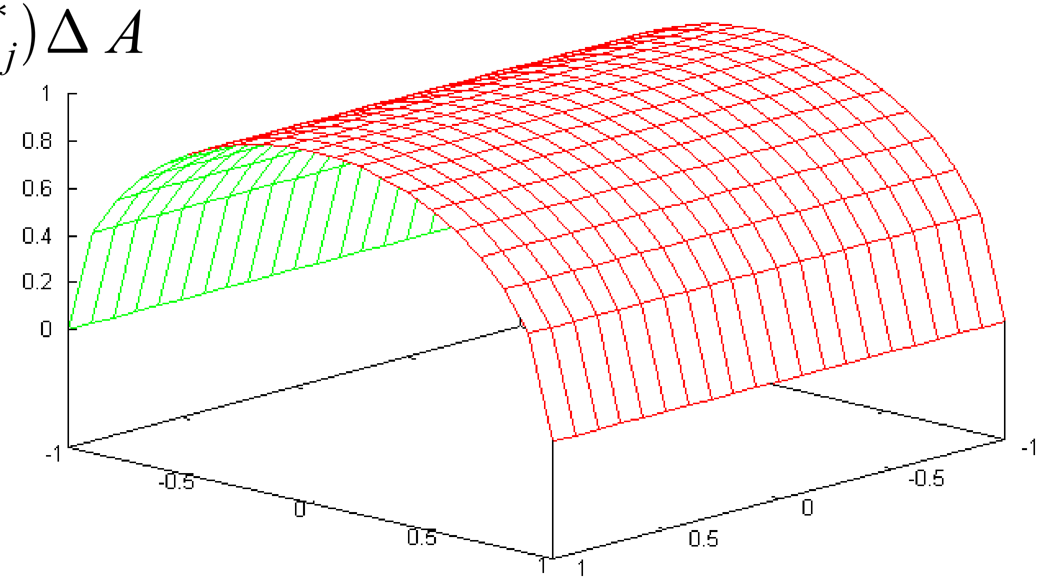
  if this limit exists.

- Note that (x_{ij}^*, y_{ij}^*) can be chosen arbitrary in each R_{ij}.

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

Double Riemann sum

- For nonnegative f, the volume V of the solid lies above the rectangle R and below the surface $f(x, y)$ is
$$V = \iint_R f(x, y)\, dA$$
- The sum is called a double Riemann sum
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$



Student note

1. Evaluate $\iint\limits_R \sqrt{1-y^2}\, dA$ on [-1, 1]×[-1, 1].

Properties of double integral

- Given that the double integral of f and g exists on R

$$\iint_R \left(f(x,y)+g(x,y)\right)dA=\iint_R f(x,y)\,dA+\iint_R g(x,y)\,dA$$

$$\iint_R c\,f(x,y)\,dA=c\iint_R f(x,y)\,dA$$

- If $f(x,y) \geq g(x,y)$ for $(x,y) \in R$.

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

Student note

1. Evaluate the double integral by first identifying it as the volume of a solid.

$$\iint_R 3\,dA = \{(x,y) \mid -2 \le x \le 2, 1 \le y \le 6\}$$

$$\iint_S (4-2y)\,dA, S = [0,1]\times[0,1]$$

Student note

2. If f is a constant function, $f(x,y) = k$ and $R = [a,b]\times[c,d]$ show that

$$\iint_R k\,dA = k(b-a)(d-c).$$

Chapter 5

5.2 Iterated integrals

Iterated integral

- Suppose f is a continuous function of two variables on $R = [a, b]\times[c, d]$.
- The notation $\int_c^d f(x, y)\,dy$ mean x is held fixed and $f(x, y)$ is *integrated with respect to y*. This defines a function of x, $A(x)=\int_c^d f(x, y)\,dy$.
- Now integrate A with respect to x from a to b we have

$$\int_a^b \int_c^d f(x, y)\,dy\,dx=\int_a^b \left(\int_c^d f(x, y)\,dy \right) dx$$

- This is called an *iterated integral*.

Student note

1. Evaluate the iterated integrals

$$\int_0^1 \int_1^2 x\, y^3 \, dy\, dx$$

$$\int_1^2 \int_0^1 x\, y^3 \, dx\, dy$$

Student note

2. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ on the first quadrant.

Fubini's Theorem

- If f is continuous on the rectangle R

$$R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Iterated integral of product

- If $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.
- Suppose $f(x, y) = g(x)\,h(y)$ on $R = [a, b]\times[c, d]$,

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Student note

1, Evaluate the double integral of $f(x,y) = x - 2y^3$ where
$R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$.

Student note

2. Evaluate the double integral of $f(x,y) = x\sin(xy)$ where $R = [0, \pi]\times[1, 2]$.

Student note

3. What is the double integral

$$\iint_R \sin(x)\cos(y)\,dA$$

where $R = [0, \pi/2]\times[0, \pi/2]$?

Student note

4. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = \{(x,y) \mid 0 \le x \le 1, -2 \le y \le 3\}$.

Student note

5. Find the volume of the solid in the first octant bounded by the cylinder $z = 6 - xy$ and the plane $x = 2$.

Chapter 5

5.3 Double integrals over general regions

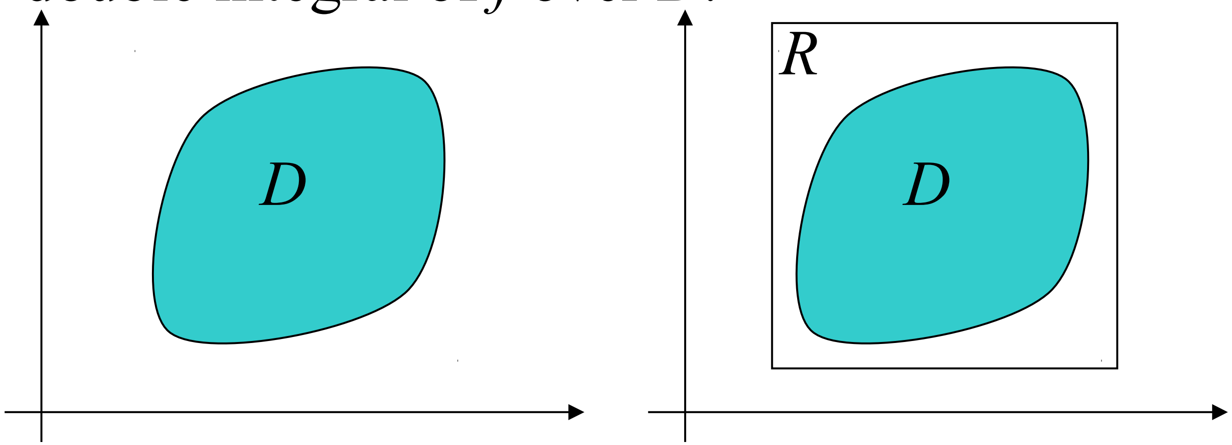
Double integral over general region

- Let f be a nonnegative on D which is enclosed in a rectangular region R.
- We define a new function F on R as

$$F(x,y)=\begin{cases} f(x,y) & if \quad (x,y)\in D \\ 0 & if \quad (x,y)\in R-D \end{cases}$$

- Then $\iint_D f(x,y)\,dA=\iint_R F(x,y)\,dA$

the double integral of f over D.



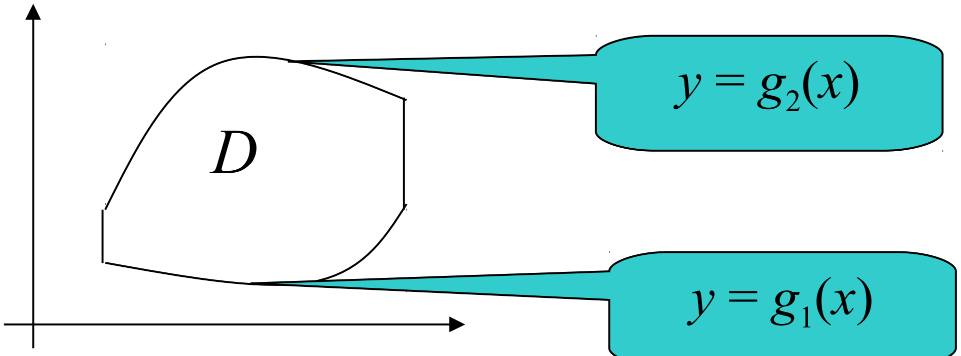
Double integral of type I

- A plane region D is said to be of *type I* if it lies between the graphs of two continuous functions of x, that is

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$



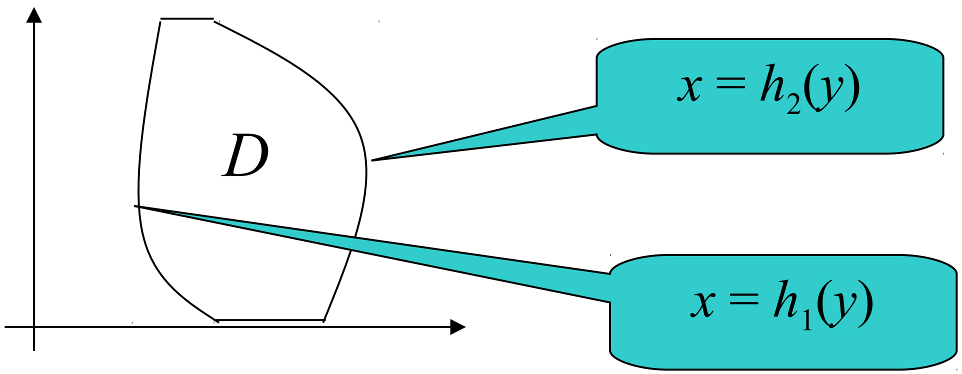
Double integral of type II

- A plane region D is said to be of *type II* if it lies between the graphs of two continuous functions of y, that is

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

Then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$



Student note

1. Evaluate the double integral of f on the given domain D_1 is bounded by the parabola $y = 2x^2$ and $y = 1 + x^2$.

$$\iint_{D_1} (2x + y)\, dA$$

Student note

2. Evaluate the double integral of f on the given domain D_2 is bounded by the line $y = x - 1$ and $y^2 = 2x + 6$.

$$\iint_{D_2} x\,y\,dA$$

Student note

3. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Student note

4. Evaluate $\int_0^1 \int_x^1 \sin\left(y^2\right) dy\, dx$.

Properties of double integral

- Given double integrals of f and g exist over region D.

$$\iint_D \left(f(x,y) + g(x,y) \right) dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$$

- If $f(x,y) \geq g(x,y)$ for $(x,y) \in D$,

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA.$$

- The area of the region D is $\iint_D 1\,dA = A(D)$.

Properties of double integral

- If $D = D_1 \cup D_2$ where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

- If $m \le f(x,y) \le M$ for all $(x,y) \in D$,

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D).$$

where $\iint_D 1\,dA = A(D).$

Student note

1. Evaluate the double integral of $f(x,y) = 2x - 3y^2$ on the domain D bounded by $y = |x| + 1$, $y = 3$.

Student note

2. Evaluate $\int_0^4 \int_{\sqrt{y}}^2 e^{x^2}\,dx\,dy.$

Student note

3. Find the area enclosed between the parabolas $y = x^2 - 4$ and $y = -x^2 + 2x$.

Student note

4. Find the volume bounded by the surface $z = 4 - x^2 - y^2$ and the plane $x + y = 1$.

Student note

5. Verify that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$.

Chapter 5

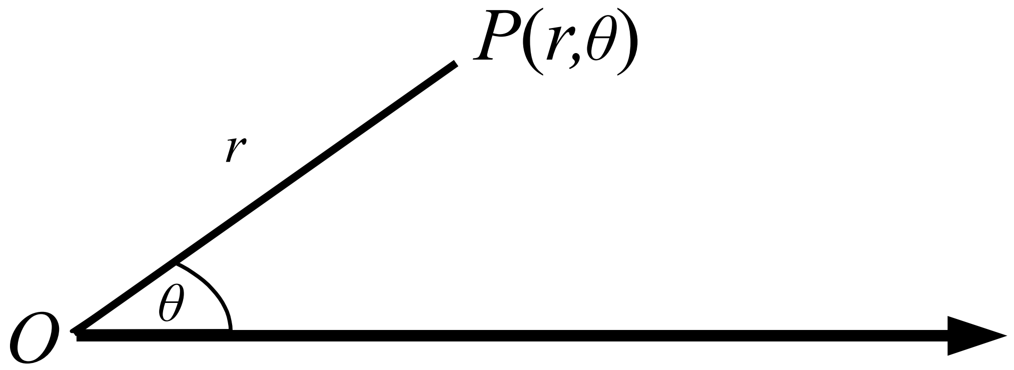
5.4 Double integrals in polar coordinates

Polar coordinates

- A cotem represents a point in the plane by an ordered pair of numbers, called **Cartesian coordinates**.
- Newton suggested another coordinate system, called the **polar coordinate system,** which is more convenient for many purposes.
- A point in the plane that is called the **pole** (or origin) and is labeled *O*.
- We draw a ray (half-line) starting at *O* called the **polar axis**. This axis usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

Polar coordinates

- If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP.



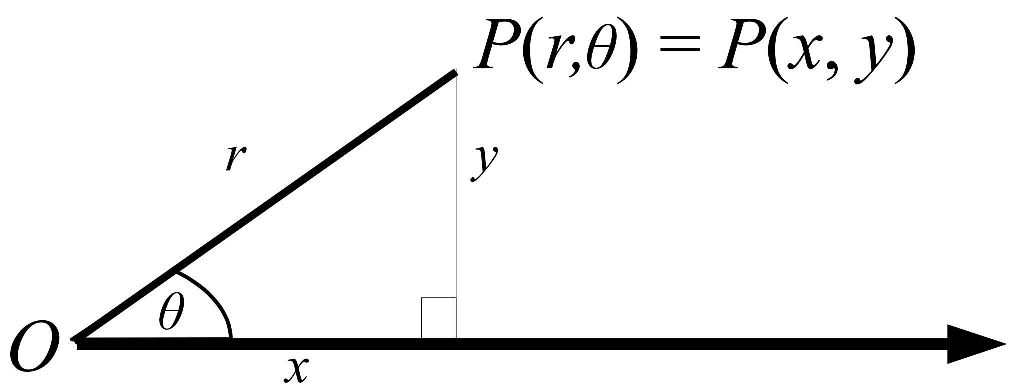
- Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P.
- If $P = O$, then r = 0 and we agree that (0, θ) represents the pole for any value of θ.

Student note

1. Plot the points whose polar coordinates are given: $P(1, 5\pi/4)$, $Q(2, 3\pi)$, $R(2, -2\pi/3)$, $S(-3, 3\pi/4)$.

Polar coordinates conversion

- The connection between polar and Cartesian coordinates can be seen from figure.



- We have $x = r\cos\theta$ and $y = r\sin\theta$.
- Moreover, $r^2 = x^2 + y^2$, $\tan\theta = y/x$.

Student note

2. Convert the point $(2, \pi/3)$ from polar to Cartesian coordinate and represent $(-1, 1)$ in terms of polar coordinates.

Polar curves

- The graph of a polar equation $r = f(\theta)$, or more generally, $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

3. Determine the curve C_1 for the polar equation $r = 2$ and C_2 for $\theta = 2$.

Student note

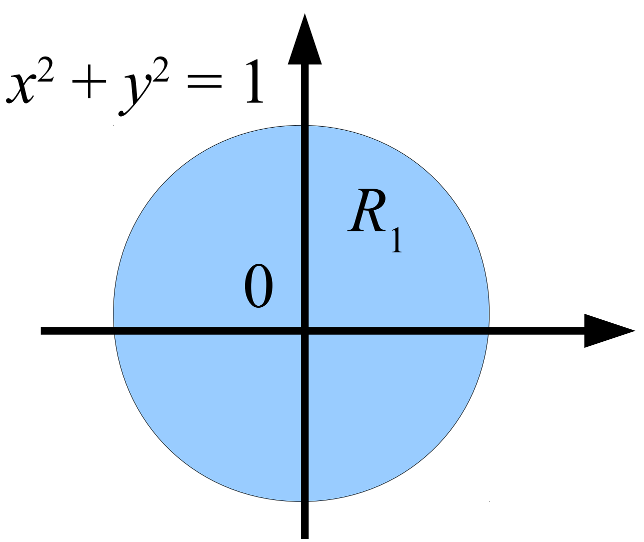
4. Sketch the curve with polar equation $r = 2\cos\theta$.

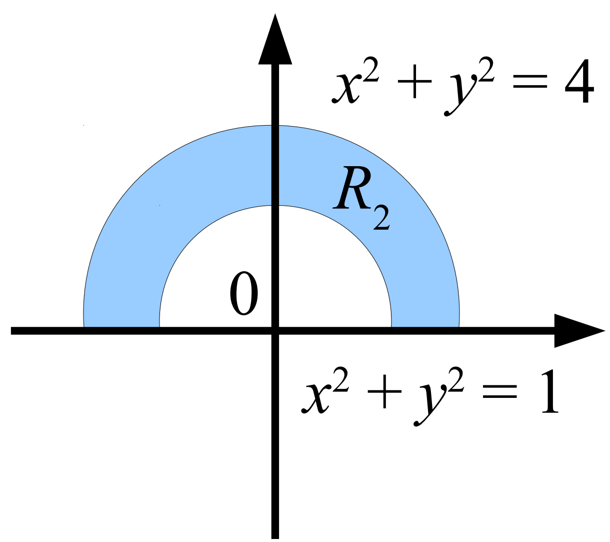
Student note

5. Sketch the curve with polar equation $r = \cos 2\theta$.

Double integrals in Polar coordinates

- If the double integral $\iint_D f(x, y)\,dA$ to evaluate on R_i as





$R_1 = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ $\quad$ $R_2 = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

Polar coordinates

- Note that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations.

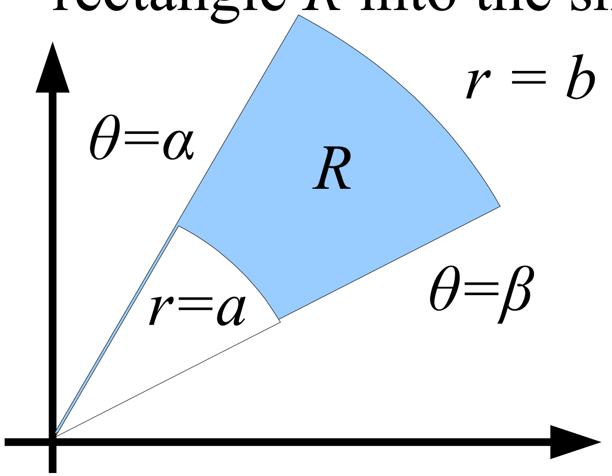
$$r^2 = x^2 + y^2, \qquad x = r\cos\theta,\ y = r\sin\theta$$

- The regions in R_1 is a special case of a polar rectangle

$$R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$$

Double integral over polar rectangle

- Divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r=(b-a)/m$
- and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta=(\beta-\alpha)/n$
- The circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles



Double integral over polar rectangle

- The "center" of the polar subrectangle

$$R_{ij} = \{(r, \theta) \mid r_{i\text{-}1} \le r \le r_i, \theta_{j\text{-}1} \le \theta \le \theta_j\}$$

- has the polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i), \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

- The area of a sector of a circle with radius r and central angle θ is $\frac{1}{2} r^2 \theta$.
- Then the area of R_{ij} is

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta$$

Double integral over polar rectangle

- For continuous function f,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

- Let $g(r, \theta) = r f(r\cos\theta, r\sin\theta)$ then

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\Delta r \Delta\theta$$

which is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta .$$

- Therefore,

$$\iint_R f(x,y)\,dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r,\theta)\, r\,dr\,d\theta .$$

Change to polar coordinates

- If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

- If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Student note

1. Evaluate the double integral of $f(x,y) = 3x + 4y^2$ on the domain D is the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Student note

2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Student note

3. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Student note

4. Verify that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$.
using a double integral in polar coordinates.

Student note

5. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx.$

Chapter 5

5.5 Applications of Double integrals

Density and mass

- A single integrals can be used to compute moments and the center of mass of a thin plate or lamina with constant density.
- The double integrals could be used to determine a lamina with variable density.
- Suppose the lamina occupies a region D of the xy-plane and its density at a point (x, y) in D is given by $\rho(x, y)$.

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

- Let m be the total mass of the lamina.

$$m = \lim_{k, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^{*}, y_{ij}^{*}) \Delta A = \iint_D \rho(x, y)\, dA.$$

Student note

1. Charge is distributed over the triangular region D so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^3). Find the total charge.

Moments and centers of mass

- Consider the center of mass of a lamina with variable density.
- Suppose the lamina occupies a region D and has density function $\rho(x, y)$.
- Define the *moment of a particle about an axis* as the product of its mass and its directed distance from the axis.
- Divide D into small rectanges. The the mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of R_{ij} with respect to the x-axis is $\rho(x_{ij}^*, y_{ij}^*)\Delta A\, y_{ij}^*$

Moments and centers of mass

- The *moment of the entire lamina about the x-axis* is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*}) \Delta A = \iint_D y\,\rho(x, y)\, dA$$

- Similarly, the *moment about y-axis* is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*}) \Delta A = \iint_D x\,\rho(x, y)\, dA$$

Define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y, m\bar{y} = M_x$.

Moments and centers of mass

- The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x, y)\,dA, \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x, y)\,dA,$$

where the mass m is given by

$$m = \iint_D x\,\rho(x, y)\,dA.$$

Student note

2. Find the mass and center of mass of a triangular lamina with vertices (0, 0), (1, 0), and (0, 2) if the density function is $\rho(x, y) = 1 + 3x + y$.

Student note

3. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Moments of inertia

- The *moment of inertia* (also called the *second moment*) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.
- We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D as

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A y_{ij}^* = \iint_D y^2 \rho(x, y)\, dA$$

- Similarly, the *moment of inertia about the y-axis* is

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A y_{ij}^* = \iint_D x^2 \rho(x, y)\, dA$$

Moment of inertia

- The *moment of inertia about the origin* also called the *polar moment of inertia* is

$$M_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left((x_{ij}^*)^2 + (y_{ij}^*)^2 \right) \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

$$= \iint_D (x^2 + y^2) \rho(x, y)\, dA$$

- Note that $I_0 = I_x + I_y$.

Student note

4. Find the moments of inertia I_x, I_y and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin and radius a.

Moment of inertia

- The *radius of gyration of a lamina about an axis* is the number R such that

$$mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis.

- In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equations

$$m\bar{\bar{y}}^2 = I_x, \quad m\bar{\bar{x}}^2 = I_y$$

Student note

5. Find the radius of gyration about the x-axis of the disk in Example 4.