Chapter 6

6[Extra] Modeling with Differential Equations

Models of population growth

- Modeling physical law based on evidence = Formulating a mathematical model.
- It takes the form of a *differential equation*, an equation that contains an unknown function and its derivatives.
  - Model for the growth of a population based on the assumption that the population grows at a rate proportional to the size of the population. (The ideal conditions for growth of bacteria, etc.)

Models of population growth

- Let t be the independent variable time and P be the number of individuals in the population.
- The rate of growth of the population is $\frac{dP}{dt}$.
- By the assumption,

$$\frac{dP}{dt} = kP$$

where k is the proportionality constant.

- This is a differential equation because it contains an unknown function P and its derivative $\frac{dP}{dt}$.

Definition

- A *differential equation* is an equation that contains an unknown function and some of its derivatives.
- The *order* of a differential equation is the order of the highest derivative that occurs in the equation.

  - $\frac{dy}{dx} = 2x^3 y + 1$
  - $y' = xy$
  - $\frac{d^3 y}{dt^3} - t\frac{dy}{dt} + (t^2 - 1)y = e^t$
  - $y''x - xy = 2$
  - $\left(\frac{d^2 y}{dx^2}\right)^3 - 8x\left(\frac{dy}{dx}\right)^4 + 3xy = x - 1$

Solution of the population growth model

- What is a solution of the population growth model?
- Same as "What function has its derivative equal to a constant multiple by itself?". Exponential function.
- If we let $P(t) = Ce^{kt}$, then

$$P'(t) = C(ke^{kt}) = kCe^{kt} = kP(t)$$

- Thus, any $P(t) = Ce^{kt}$ is a solution of the population growth model.
- Varying C, we get the *family* of solutions $P(t) = Ce^{kt}$.

Solution of a differential equation

- A function f is called a *solution* of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.
- If f is a solution of $y' = xy$, then $\forall x \in I$,

$$f'(x) = xf(x).$$

- To *solve* a differential equation means finding all possible solutions of the equation.

The solution of $y' = x$ is $y = \dfrac{x^2}{2} + C$.

Student note

1. Show that every member of the family of functions

$$y=\frac{1+ce^x}{1-ce^x}$$

is a solution of the differential equation $y'=\frac{y^2-1}{2}$.

Initial condition

- Usually, we are not interested in a family of solutions (*general solution*). We want to find a particular solution that satisfies some additional requirement.
- For example, it must satisfy a condition $y(t_0) = y_0$.
- This is called an *initial condition* and the problem of finding a solution of this is called an initial-value problem.

Student note

2. Find a solution of the differential equation $y' = \dfrac{y^2 - 1}{2}$ that satisfies the initial condition $y(0) = 2$.

General and specific solution

- Consider

$$\frac{dy}{dx} - y = e^{2x}$$

then $y = e^{2x}$ is a solution of this differential equation.

- In addition, $y = e^x + e^{2x}$ is also one of the solution of this differential equation.
  - $y = Ce^x + e^{2x}$ where C is a real number is a *general solution* of the differential equation.
  - $y = e^x + e^{2x}$ is a *specific solution* for the differential equation.

Student note

3. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$.

Student note

4. For what values of r does the function $y = e^{-rt}$ satisfy the differential $y'' + y' + 6y = 0$.

Student note

5. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2$$

5.1 What are the constant solutions of the equation?

5.2 For what values of y is y increasing?

5.3 For what values of y is y decreasing?

Chapter 6

6.1. Separable equations

Separable equations

- A *separable equation* is a first-order differential equation in which the expression y' can be factored as a function of x times the function of y. That means

$$\frac{dy}{dx} = g(x)f(y)$$

- We rewrite $h(y) = \frac{1}{f(y)}$. It becomes

$$h(y)\,dy = g(x)\,dx$$

- So that all *y's* are on one side of the equation and all *x's* are on the other side

$$\int h(y)\,dy = \int g(x)\,dx$$

Verification

- We can verify that $\int h(y)\,dy = \int g(x)\,dx$ is the solution of $\dfrac{dy}{dx} = g(x)f(y)$
- By using the Chain rule,

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\frac{dy}{dx} = g(x)$$

$$h(y)\frac{dy}{dx} = g(x)$$

Student note

1. Solve the differential equation $\frac{dy}{dx} = \frac{x}{y^2}$ and find the solution with the initial condition $y(0) = 1$.

Student note

2. Solve the differential equation $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$.

Student note

3. Solve the differential equation $\dfrac{dy}{dx}=\dfrac{3x^3}{y-\sin y}$.

Student note

4. Solve the equation $y' = xy$.

Chapter 6

6.1[Extra] Exponential growth and decay

Exponential growth and decay

- The assumption that the population grows at a rate proportional to the size of the population:
  - If $P = 100$ and growing rate $P' = 30$.
  - Another bacterial of the same type with 100 more would have growing rate 60.
- The same assumption applies to
  - (nuclear physics) the mass of a radioactive substance decays at a rate proportional to the mass.
  - (chemistry) the rate of a unimolecular first-order reaction is proportional to the concentration of the substance.

Law of natural growth/decay

- In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This is called the *law of natural growth* for $k > 0$ or the *law of natural decay* for $k < 0$.

- Hence, $\displaystyle\int \frac{1}{y}\frac{dy}{dt} = \int k\,dt$

$$\ln|y| = kt + C$$

$$y = Ae^{kt}$$

For $t = 0$, $y(0) = Ae^{k0} = A$.

Population growth

- The solution of the initial-value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is
$$y(t) = y_0 e^{kt}$$

- The quantity $\frac{1}{P}\frac{dP}{dt}$ is called the *relative growth rate*.
- Hence, "the growth rate is proportional to population size" is the same as "the relative growth rate is constant."

Student note

1. A bacteria culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.
   1.1 Find the initial population.
   1.2 Find an expression for the population.
   1.3 Find the number of cells after 5 hours.
   1.4 Find the rate of growth after 5 hours.
   1.5 When will the population reach 200,000?

Radioactive decay

- Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance at after time t,

$$-\frac{1}{m}\frac{dm}{dt}$$

- Hence, $\dfrac{dm}{dt} = km \equiv m(t) = m_0 e^{kt}$.
- Physicists express the rate of decay in terms of *half-life*, the time required for half of any given quantity.

Student note

2. The half-life of radium-226 $\left({}^{226}_{88}Ra\right)$ is 1590 years.

   2.1 A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of ${}^{226}_{88}Ra$ that remains after t years.

   2.2 Find the mass after 1000 years correct to the nearest milligram.

   2.3 When will the mass be reduced to 30 mg?

Newton's law of cooling

- The rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings,

$$\frac{dT}{dt} = k(T - T_s)$$

Student note

4. A bottle of soda pop at room temperature (72ºF) is placed in a refrigerator where the temperature is 44ºF. After half an hour the soda pop has cooled to 61ºF.
   4.1 What is the temperature of the soda pop after another half hour?
   4.2 How long does it take for the soda pop to cool to 50ºF?

Continuously compounded interest

- If \$1000 is invested at 6% interest, compounded annually,
  - after 1 year the investment is worth \$1000(1.06)=\$1060
  - after 2 years it's worth \$1060×\$1000(1.06)=\$1123.60
  - after t year it's worth \$1000$(1.06)^t$.
- In general, if A_0 is an initial investment and r is an interest rate, then after t years it's worth

$$A_0(1+r)^t$$

- Usually, interest is compounded more frequently, n times a year,

$$A_0\left(1+\frac{r}{n}\right)^{nt}$$

- Or continuously, $A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$

Student note

5. If \$500 is borrowed at 14% interest, find the amounts due at the end of 2 years if the interest is compounded.

   5.1 annually

   5.2 quarterly

   5.3 monthly

   5.4 daily

   5.5 hourly

   5.6 continuously

Chapter 6

6.2 Homogeneous differential equations

Homogeneous function

- A function of two variables, x and y is called the *homogeneous function* of degree n if for any $t > 0$,

$$f(tx, ty) = t^n f(x, y)$$

- The first order *homogeneous differential equation* is of the form

$$M(x,y)dx + N(x,y)dy = 0$$

where $M(x,y)$, $N(x,y)$ are homogeneous function of the same degree.

Homogeneous equations

- Given $M(x,y)$, $N(x,y)$ as the homogeneous function of degree n,

$$M(tx,ty) = t^n M(x,y),\ N(tx,\ ty) = t^n N(x,y),$$

- We can transform $M(x,y)dx + N(x,y)dy = 0$ using $v=y/x$, where $t = 1/x$ for positive value of x,

$$x^n M(1,v)dx + x^n N(1,v)dy = 0$$

$$M(1,v)dx + N(1,v)(vdx+xdv) = 0$$

$$(M(1,v) + vN(1,v))dx + xN(1,v)dv = 0$$

- This differential equation is just a separable differential equation

$$\frac{1}{x}dx+\frac{N(1,v)}{M(1,v)+vN(1,v)}dv=0.$$

Student note

1. Find the general solution of $(x - y)dx = (x + y)dy$.

Student note

2. Find the general solution of $(x^3+xy^2)dy = 2y^3dx$.

Student note

3. Find the general solution of $(xe^{y/x}+y)dx - xdy = 0$.

Student note

4. Find the specific solution of $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos\left(\dfrac{y-x}{x}\right)$ when $y(2) = 2$.

Student note

5. Find the specific solution for $y(2) = \pi$

$$\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)dx + x\cos\left(\frac{y}{x}\right)dy = 0.$$

Chapter 6

6.3 Exact differential equations

Exact differential equation

- The differential equation $M(x,y)dx + N(x,y)dy = 0$ is the *exact differential equation* if there exists $f(x,y)$ that

$$df(x,y) = M(x,y)dx + N(x,y)dy$$

- Then the general solution is $f(x,y) = c$.
- It is possible to determine exactness and find the function f by mere inspection, for example

$$y\,dx + x\,dy = 0$$

Student note

1. Determine the solution of $2xy\,dx + x^2 dy = 0$.

Student note

2. Determine the solution of $(y^3 - 2x)dx + (3xy^2 - 1)dy = 0$.

Student note

3. Determine the solution of $\dfrac{dy}{dx} = -\dfrac{ax+by}{bx+cy}$.

Theorem for exact equation

- Theorem: If $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ is continuous on a rectangle R then $M(x,y)dx + N(x,y)dy = 0$ is the exact differential equation if and only if

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

- Note that

$$dF(x,y) = M(x,y)dx + N(x,y)dy$$

$$\frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y) = M(x,y)dx + N(x,y)dy$$

Solution of the exact equation

- Then

$$F(x,y)=\int \frac{\partial F}{\partial x}(x,y)\,dx=\int M(x,y)\,dx=g(x,y)+k(y)$$

$$\frac{\partial F}{\partial y}(x,y)=N(x,y)$$

- Hence,

$$N(x,y)=\frac{\partial}{\partial y}g(x,y)+k'(y)$$

$$k(y)=\int \left(N(x,y)-g(x,y)\right)dy$$

Student note

1. Determine the general solution of $(y^3 - 2x)dx + (3xy^2 - 1)dy = 0$.

Student note

2. Determine the general solution of $(x^2+y^2)dx + (2xy+\cos y)\,dy = 0$.

Student note

3. Determine the general solution of $y \cos xy\, dx + x \cos xy\, dy = 0$.

Student note

4. Find the general solution of $(2xe^y+e^x)dx + (x^2+1)e^y dy = 0$.

Student note

5. Determine the general solution of

$$\left(e^{x}+\ln y+\frac{y}{x}\right)dx+\left(\frac{x}{y}+\ln x+\sin y\right)dy=0.$$

Chapter 6

6.4 Integrating factor

Integrating factors

- Consider the equation

$$y\,dx + (x^2y - x)\,dy = 0$$

  is easily seen to be nonexact, since $\dfrac{\partial M}{\partial y} = 1, \dfrac{\partial N}{\partial x} = 2xy - 1.$

- However, if we multiply through by the factor $1/x^2$

$$\frac{y}{x^2}\,dx + \left(y - \frac{1}{x}\right)dy = 0$$

  which is exact.

- Under what conditions can a function $\mu(x, y)$ be found with the property that $\mu(M\,dx + N\,dy) = 0$ (**) is exact.

Integrating factors

- Assume that (**) has a general solution

$$f(x, y) = c$$

by differentiating

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} = 0$$

Hence,

$$\frac{dy}{dx} = -\frac{M}{N} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

So

$$\frac{\frac{\partial f}{\partial x}}{M} = \frac{\frac{\partial f}{\partial y}}{N}$$

Integrating factors

- Denote the common ratio by $\mu(x, y)$ then

$$\frac{\partial f}{\partial x} = \mu M, \frac{\partial f}{\partial y} = \mu N$$

- From $M(x, y)\, dx + N(x, y)\, dy = 0$, multiply by $\mu(x, y)$

$$\mu M(x, y)\, dx + \mu N(x, y)\, dy = 0,$$

So

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

is exact.

- From Clairaut's theorem, we assume continuity of both f_{xy} and f_{yx}, then $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Integrating factors

- That means we need

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\frac{1}{\mu}\left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} \right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

- If μ is a function of x alone, then

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$$

Theorem

- Given $M(x,y)dx + N(x,y)dy = 0$, a differential equation has the integrating factor, $I(x,y)$
  - if $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$ is just a function of x.
    This case has the integrating factor as
    $$I(x,y)=e^{\int f(x)dx}$$
  - if $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=g(y)$ is just a function of y.
    This case has the integrating factor as
    $$I(x,y)=e^{\int g(y)dy}$$

Student note

1. Solve the differential equation $y' + 2y = e^{3x}$.

Student note

2. Solve the differential equation $y' + y \tan x = 2 \sin x$.

Student note

3. Find the solution of $(x^2y^2 + y)\,dx - x\,dy = 0$ when $y(2) = -1$.

Student note

4. Find the solution of $\dfrac{dy}{dx} = \dfrac{2xy + y^2}{y - 2xy - x^2}$.

Student note

5. Solve the differential equation $\frac{2x-y}{x^2+y^2}\,dx+\frac{x+2y}{x^2+y^2}\,dy=0.$

Chapter 6

6.5 Linear differential equations

A first order linear differential equation

- A first order linear differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

  where P and Q are continuous functions on I.
- Consider the first order linear differential equation

$$xy' + y = 2x$$

- LHS: $xy' + y = (xy)'$ RHS: $\int 2x\,dx = 2\frac{x^2}{2} + C$
- Therefore, $xy = x^2 + C$.

Student note

1. Find the general solution of $y' + \frac{1}{x} y = 3.$

Integrating factor

- To solve the first order linear differential equation, we multiply the suitable integrating factor, $I(x)$.

$$\frac{dy}{dx}+P(x)y=Q(x)$$

  to get

$$I(x)\left(\frac{dy}{dx}+P(x)y\right)=I(x)Q(x).$$

- If we can find such a function $I(x)$,

$$(I(x)y)'=I(x)Q(x).$$

- Hence,

$$I(x)y=\int I(x)Q(x)dx+C.$$

Integrating factor

- The property of $I(x)$ is

$$I(x)P(x) = I'(x).$$

- This is a separable differential equation,

$$\int \frac{1}{I(x)} dI(x) = \int P(x)dx$$

$$\ln|I(x)| + C = \int P(x)dx$$

- Hence,

$$I(x) = Ae^{\int P(x)dx}.$$

Student note

1. Solve the differential equation $\frac{dy}{dx}+3x^2y=6x^2$.

Student note

2. Find the solution of the initial-value problem

   $x^2y' + xy = 1$, $x > 0$ and $y(1) = 2$.

Student note

3. Solve $y' + 2xy = 1$.

Student note

4. Solve the initial-value problem $y' - y = 2xe^{2x}$, $y(0) = 1$.

Student note

5. Solve the initial-value problem $xy' + 2y = \sin x$ and

$$y\left(\frac{\pi}{2}\right) = 1.$$

Bernoulli differential equation

- A *Bernoulli differential equation* is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

if $n = 0$ or $n = 1$, the Bernoulli differential equation is just linear.

- We can easily transform this differential equation to linear by $u = y^{1-n}$ then

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Student note

1. Solve the differential equation $y' + \dfrac{y}{x} = y^3$.

Student note

2. Solve the differential equation $2\frac{dy}{dx} + xy = x^3 y^2$.

Student note

3. Find the solution of the initial-value problem

$$\frac{dy}{dx} - y\cot x = y^2 sec^2 x \qquad \text{and} \quad y\left(\frac{\pi}{4}\right) = -1.$$