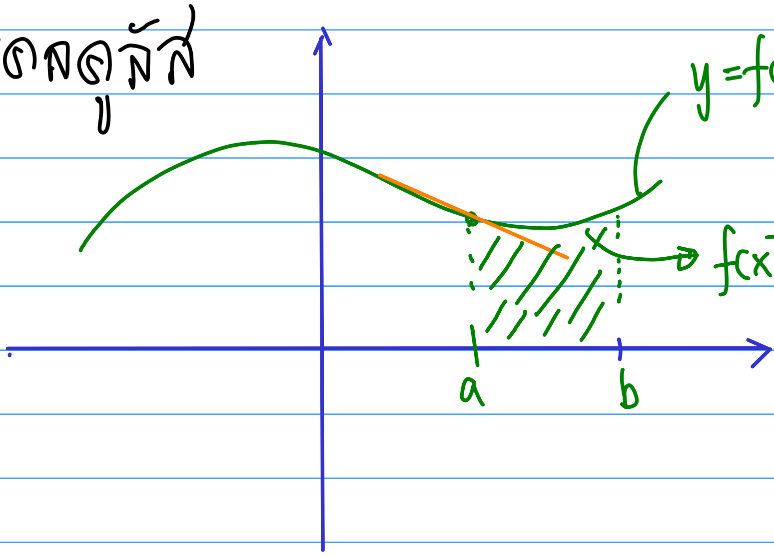
แคลคูลัส

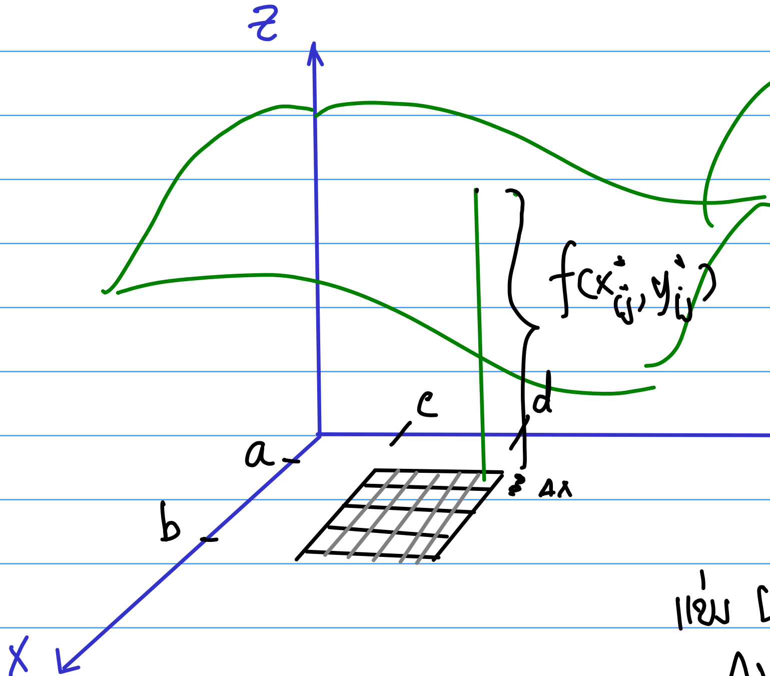
อินทิกรัล



$y = f(x)$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$f(x) \geq 0$



$z = f(x,y)$

$f(x,y) \geq 0 \quad \forall (x,y)$ บน D

$D = [a,b] \times [c,d]$

แบ่ง $[a,b]$ ออกเป็น n ช่วงย่อยเท่าๆกัน

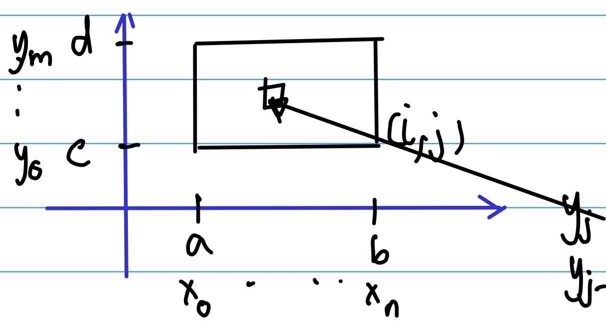
$\Delta x = \dfrac{b-a}{n}$

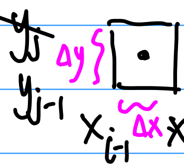
$$\begin{array}{ccccccccccccc} x_0 & < & x_1 & < & x_2 & < & \dots & < & x_i & < & \dots & < & x_n \\ \shortparallel & & \shortparallel & & \shortparallel & & & & & & & & \shortparallel \\ a & & a+\Delta x & & a+2\Delta x & & & & a+i\Delta x & & & & b \end{array}$$

แบ่ง $[c,d]$ ออกเป็น m ช่วงย่อยเท่าๆกัน

$\Delta y = \dfrac{d-c}{m}$

$$\begin{array}{ccccccccccccc} y_0 & < & y_1 & < & y_2 & < & \dots & < & y_j & < & \dots & < & y_m \\ \shortparallel & & \shortparallel & & \shortparallel & & & & & & & & \shortparallel \\ c & & c+\Delta y & & c+2\Delta y & & & & c+j\Delta y & & & & d \end{array}$$





ให้ $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

ปริมาตรของกล่อง i,j $\quad f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$

$$\iint_D f\,dA = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$$

เทคนิคอินทิเกรต

1. $\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$

ต.ย. $\int x\sqrt{x^2+1}\,dx = \int \sqrt{u}\,\frac{du}{2} = \frac{2u^{\frac{3}{2}}}{3\times 2}+C = \frac{(x^2+1)\sqrt{x^2+1}}{3}+C$

$u = x^2+1$, $du = 2x\,dx$

↳ ดิฟเฟอเรนเชียล

FOC, $\frac{d}{dx}\left(\frac{(x^2+1)^{\frac{3}{2}}}{3}+C\right) = \frac{\cancel{3}}{2\times\cancel{3}}(x^2+1)^{\frac{1}{2}}\frac{d}{dx}(x^2+1)+0 = x\sqrt{x^2+1}$

2. $\int u\,dv = uv - \int v\,du$

ต.ย. $\int \underset{u}{\ln x}\,\underset{dv}{dx} = x\ln x - \int \cancel{x}\,\frac{1}{\cancel{x}}\,dx = x\ln x - x + C$

$\therefore v = x$

FOC, $\frac{d}{dx}(x\ln x - x + C) = \frac{x}{x} + \ln x - 1 + 0 = \ln x$

3. $\int \frac{P(x)}{Q(x)}\,dx$ พหุนาม $Q(x)$ แยกตัวประกอบ

$b^2-4ac \geq 0$

$ax^2+bx+c = 0$ ก็ต่อเมื่อ $x = \underset{r_1}{\frac{-b+\sqrt{b^2-4ac}}{2a}}$, $x = \underset{r_2}{\frac{-b-\sqrt{b^2-4ac}}{2a}}$

$a(x-r_1)(x-r_2) = 0$

$x^3 - 1 = (x-1)(x^2+x+1)$

4. $\int \sin^m(x)\cos^n(x)\,dx$

5. $\int P(x)\,dx$

$P(x) = \sqrt{a^2-x^2}$, $x = a\sin\theta$

$P(x) = \sqrt{a^2+x^2}$, $x = a\tan\theta$

$P(x) = \sqrt{x^2-a^2}$, $x = a\sec\theta$

$$\iint_D f\,dA = \int_c^d \underbrace{\boxed{\int_a^b f(x,y)\,dx}}_{}\,dy = \int_a^b \boxed{\int_c^d f(x,y)\,dy}\,dx$$

y คงที่   x คงที่

การอินทิเกรตซ้อน

$D = [a,b]\times[c,d]$

ต.ย. ให้ $z = f(x,y) = x+y$ บน $[0,1]\times[0,1]$

$$\int_0^1\int_0^1 (x+y)\,dx\,dy = \int_0^1 \frac{x^2}{2}\Big|_{x=0}^{x=1} + yx\Big|_{x=0}^{x=1}\,dy = \int_0^1 \frac{1}{2}+y\,dy = \frac{1}{2}y\Big|_{y=0}^{y=1} + \frac{y^2}{2}\Big|_{y=0}^{y=1} = 1$$

$$\int_0^1\int_0^1 (x+y)\,dy\,dx = \int_0^1 xy\Big|_{y=0}^{y=1} + \frac{y^2}{2}\Big|_{y=0}^{y=1}\,dx = \int_0^1 x+\frac{1}{2}\,dx = \frac{x^2}{2}\Big|_{x=0}^{x=1} + \frac{1}{2}x\Big|_{x=0}^{x=1} = 1$$

แบ่ง $[0,1]$ ออกเป็น n ช่วงย่อยเท่าๆ กัน $\Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}\ \forall i\in\{0,\dots,n\}$

แบ่ง $[0,1]$ ออกเป็น m ช่วงย่อยเท่าๆ กัน $\Delta y = \frac{1}{m}$, $y_j = \frac{j}{m}\ \forall j\in\{0,\dots,m\}$

เลือก $(x^*_{ij}, y^*_{ij}) = (x_i, y_j) \in [x_{i-1},x_i]\times[y_{j-1},y_j]$

$$\iint_D f\,dA = \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x^*_{ij},y^*_{ij})\,\Delta x\,\Delta y \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i+y_j)\frac{1}{n}\frac{1}{m}$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\frac{1}{m}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{i}{n}+\frac{j}{m}\right)\right)$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\frac{1}{m}\left(\sum_{i=1}^{n}\left(\frac{i}{n}m + \frac{1}{m}\sum_{j=1}^{m} j\right)\right) \qquad \frac{1}{2}+\frac{1}{2} = 1$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\frac{1}{m}\left(\frac{m}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} + \frac{n}{\cancel{m}}\frac{\cancel{m}(m+1)}{2}\right) = \lim_{m\to\infty}\lim_{n\to\infty}\left(\left(\frac{n+1}{2n}\right)+\left(\frac{m+1}{2m}\right)\right)$$