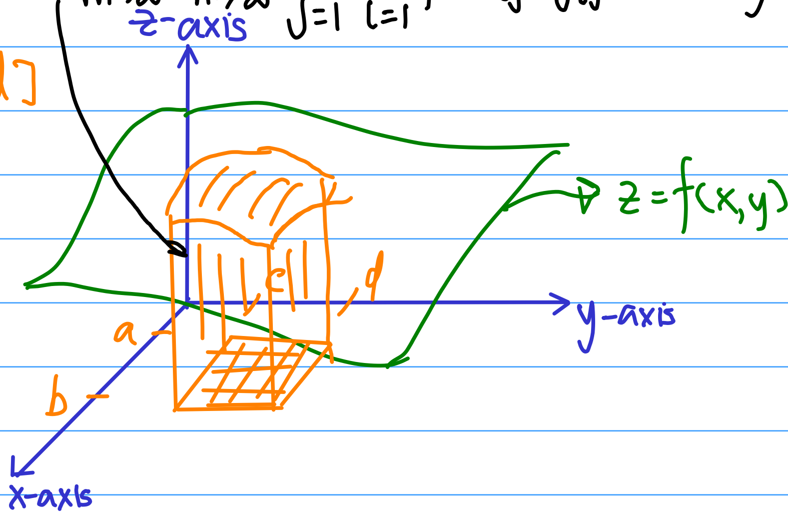
แคลคูลัส

$f(x,y) \geq 0 \ \forall (x,y) \in R$

$$\iint_R f\, dA = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x^*_{ij}, y^*_{ij})\, \Delta x\, \Delta y$$ ปริมาตรของรูปทรงตัน

$R = [a,b] \times [c,d]$

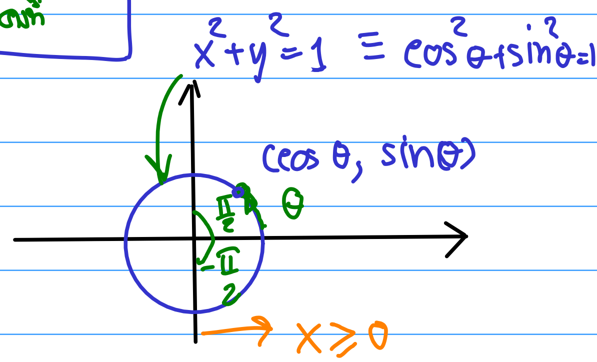


กำหนดให้ $z = f(x,y)$ และ $R = [a,b] \times [c,d]$ สี่เหลี่ยมผืนผ้า

ทบ. $$\iint_R f\, dA = \int_c^d \underbrace{\int_a^b f(x,y)\, dx}_{y \text{ คงที่}}\, dy = \int_a^b \underbrace{\int_c^d f(x,y)\, dy}_{x \text{ คงที่}}\, dx$$

ต.ย. $\iint_R \sqrt{1-y^2}\, dA$ บน $R = [-1,1] \times [-1,1]$

วิธีทำ $$\iint_R \sqrt{1-y^2}\, dA = \int_{-1}^{1} \int_{-1}^{1} \sqrt{1-y^2}\, dy\, dx$$



(trigonometric substitution)

$\int_{-1}^{1} \sqrt{1-y^2}\, dy$ ให้ $y = \sin\theta$, $dy = \cos\theta\, d\theta$, $\sqrt{1-y^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

ถ้า $y = -1$, $-1 = \sin\theta$ ได้ว่า $\theta = -\frac{\pi}{2}$

ถ้า $y = 1$, $1 = \sin\theta$ ได้ว่า $\theta = \frac{\pi}{2}$

$$\int_{-1}^{1} \sqrt{1-y^2}\, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$ trigonometric integration

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - 1 + \cos^2\theta$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ (เมื่อ } n \neq -1)$$

$$\cos^2\theta = \frac{\cos(2\theta)+1}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(2\theta)}{2} + \frac{1}{2}\,d\theta = \frac{1}{2\times 2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta)\,d(2\theta) + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta$$

$$= \frac{1}{4}\sin(2\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} + \frac{1}{2}\theta\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}$$

$$= 0 + \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$$

$$\iint_R \sqrt{1-y^2}\,dA = \int_{-1}^{1}\int_{-1}^{1}\sqrt{1-y^2}\,dy\,dx = \int_{-1}^{1}\frac{\pi}{2}\,dx = \frac{\pi}{2}\,x\Big|_{x=-1}^{x=1} = \frac{\pi}{2}(1-(-1)) = \pi$$

$$= \int_{-1}^{1}\underbrace{\int_{-1}^{1}\sqrt{1-y^2}\,dx}_{y \text{ คงที่}}\,dy = \int_{-1}^{1}\sqrt{1-y^2}\;x\Big|_{x=-1}^{x=1}dy = 2\int_{-1}^{1}\sqrt{1-y^2}\,dy = 2\times\frac{\pi}{2} = \pi$$

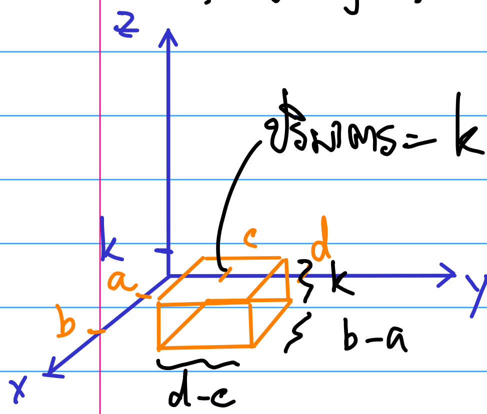
ทบ.

$$\iint_R f(x,y) + g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R f(x,y) - g(x,y)\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$$

$$\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$$

ถ้า $f(x,y) \geq g(x,y)\ \forall (x,y) \in R$ แล้ว $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$



ปริมาตร = $k(b-a)(d-c)$

$R = [a,b]\times[c,d]$

$$\iint_R k\,dA = \int_c^d\int_a^b k\,dx\,dy = k\int_c^d x\Big|_{x=a}^{x=b}dy$$

$$= k(b-a)\;y\Big|_{y=c}^{y=d} = k(b-a)(d-c)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{เมื่อ } n = -1$$

$$\int_0^1\int_1^2 xy^3\,dy\,dx = \int_1^2\int_0^1 xy^3\,dx\,dy = \int_1^2 y^3 \frac{x^2}{2}\Big|_{x=0}^{x=1} dy = \frac{1}{2}\frac{y^4}{4}\Big|_{y=1}^{y=2} = \frac{15}{8}$$

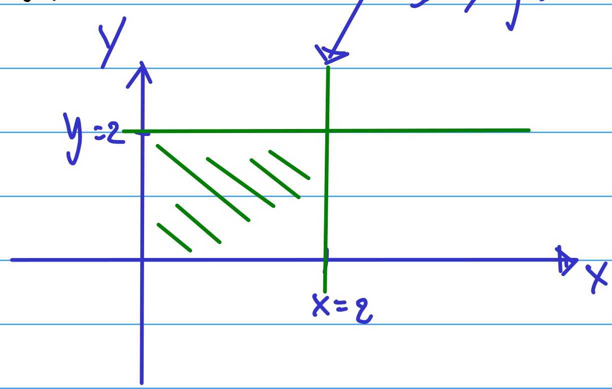
$$\int_0^1 x \frac{y^4}{4}\Big|_{y=1}^{y=2} dx = \frac{15}{4}\int_0^1 x\,dx = \frac{15}{4}\left(\frac{x^2}{2}\Big|_{x=0}^{x=1}\right) = \frac{15}{8}$$

องศา1 พาราโบลา

ต.ย. จงหาปริมาตรของรูปทรงตัน ของ $x^2+2y^2+z=16$ (Elliptic paraboloid) กับระนาบ $x=2$, $y=2$, บนออคแทนต์แรก

z ตัวที่ x,y ตัวที่

วิธีทำ วาดโดเมน $x \geq 0,\ y \geq 0$



สรุป $R = [0,2]\times[0,2] = \{(x,y)\,|\,0 \leq x \leq 2,\ 0 \leq y \leq 2\}$

$$\iint_R z\,dA = \int_0^2\int_0^2 16 - x^2 - 2y^2\,dx\,dy$$

$$= \int_0^2 16x\Big|_{x=0}^{x=2} - \frac{x^3}{3}\Big|_{x=0}^{x=2} - 2y^2 x\Big|_{x=0}^{x=2} dy$$

$$= 32y\Big|_{y=0}^{y=2} - \frac{8}{3}y\Big|_{y=0}^{y=2} - 4\frac{y^3}{3}\Big|_{y=0}^{y=2}$$

$$= 64 - \frac{16}{3} - \frac{32}{3} = 64 - 19 = 45$$

ต.ย. $\iint_R x\sin(xy)\,dA$ เมื่อ $R = [0,\pi]\times[1,2]$

วิธีทำ $\iint_R x\sin(xy)\,dA = \int_0^\pi\int_1^2 x\sin(xy)\,dy\,dx$

$$\int_1^2 \underbrace{x}_{x \text{ คงที่}}\sin(xy)\,dy = \int_x^{2x} \sin(u)\,du = -\cos(u)\Big|_{u=x}^{x=2x} = -\cos(2x) + \cos(x)$$

โดเมนที่ไม่ใช่ช่วง

$u = xy \Rightarrow du = x\,dy$

ถ้า $y=1$ แล้ว $u=x$ และ ถ้า $y=2$ แล้ว $u=2x$

$$\iint_R x\sin(xy)\,dA = -\frac{1}{2}\int_0^\pi \cos(2x)\,d(2x) + \int_0^\pi \cos(x)\,dx$$

$$= -\frac{1}{2}\sin(2x)\Big|_{x=0}^{x=\pi} + \sin(x)\Big|_{x=0}^{x=\pi}$$

$$= -\frac{1}{2}\times 0 + 0 - 0 = 0$$

$$\int_1^e\int_0^{\frac{\pi}{2}} \frac{y\cos(y\ln x)}{x}\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_1^e \frac{y\cos(y\ln x)}{x}\,dx\,dy$$

$$\int \underbrace{y}_{u}\,\underbrace{\cos(ky)\,dy}_{dv} \qquad \frac{1}{x}\cos(\ln x)\,dx = \cos(\ln x)\,d\ln x$$

$u = y\ln x$, $du = y\frac{1}{x}dx$ ถ้า $x=e$ ได้ $u=y$; ถ้า $x=1$ ได้ $u=0$

y คงที่

$$y\int_1^e \frac{\cos(y\ln x)}{x}\,dx = \int_0^y \cos(u)\,du = \sin(u)\Big|_{u=0}^{u=y} = \sin(y) - 0$$

$$\int_0^{\frac{\pi}{2}}\int_1^e y\frac{\cos(y\ln x)}{x}\,dx\,dy = \int_0^{\frac{\pi}{2}}\sin(y)\,dy = -\cos(y)\Big|_{y=0}^{y=\frac{\pi}{2}} = -(0-1) = 1$$

$$\int_1^e\int_0^{\frac{\pi}{2}} \frac{y\cos(y\ln x)}{x}\,dy\,dx$$

$$\int u\,dv = uv - \int v\,du$$

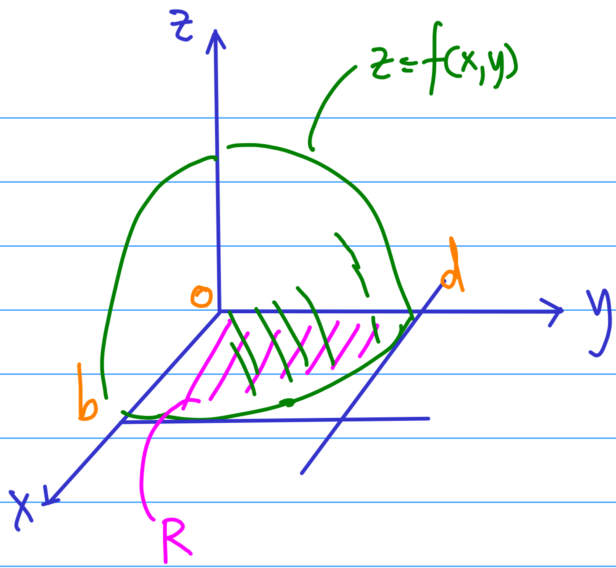
y คงที่

$u = y$ และ $dv = \frac{\cos(y\ln x)}{x} \Rightarrow \int \frac{\cos(y\ln x)}{x}\,dx = \frac{1}{y}\int \cos(y\ln x)\,d(y\ln x)$

$$= \frac{1}{y}\sin(y\ln x)$$

$$\int u\,dv = uv - \int v\,du = y\frac{1}{y}\sin(y\ln x) - \int \frac{1}{y}\sin(y\ln x)\,dy$$

พิสูจน์ ปริมาตรของรูปทรงกลมรัศมี r คือ $\frac{4}{3}\pi r^3$



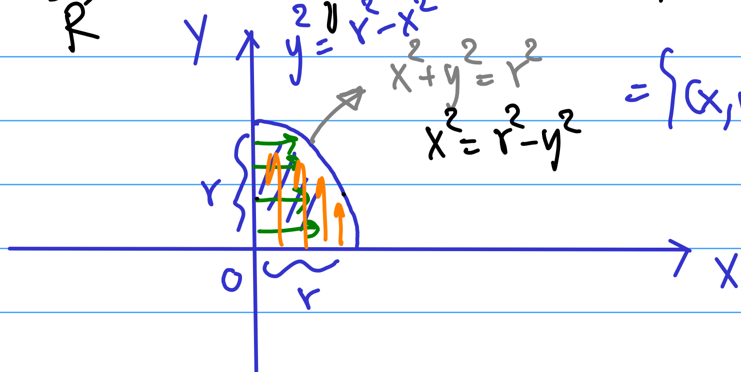
$$\tilde{f}(x,y)=\begin{cases} f(x,y) & \text{ถ้า } (x,y)\in R \\ 0 & \text{ถ้า } (x,y)\notin R \end{cases}$$

$$\iint_R f(x,y)\,dA=\int_0^b\int_0^d \tilde{f}(x,y)\,dy\,dx$$

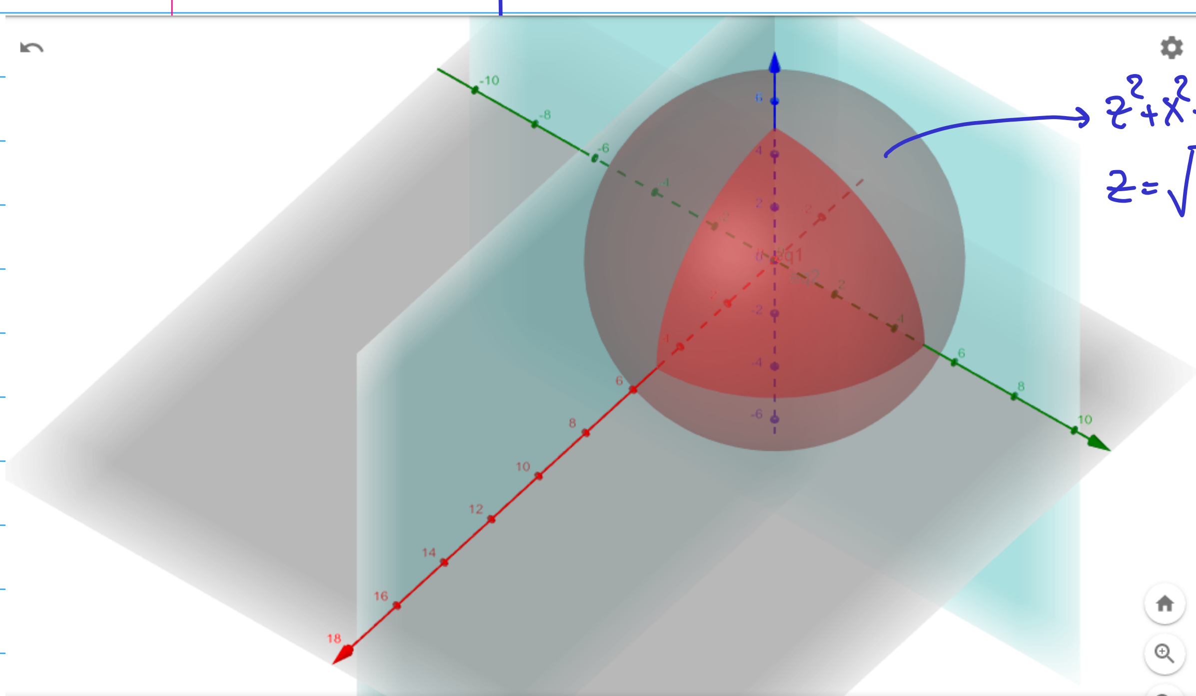
อ๊ธ

$\iint_R f\,dA$ ดูโดเมน $D=\{(x,y) \mid \underset{\text{ซ้าย}}{0}\le x\le \underset{\text{ขวา}}{\sqrt{r^2-y^2}},\ 0\le y\le r\}$

$=\{(x,y) \mid 0\le y\le \sqrt{r^2-x^2},\ 0\le x\le r\}$



$$\iint_R f\,dA=\int_0^r\int_0^{\sqrt{r^2-x^2}}\sqrt{r^2-x^2-y^2}\,dy\,dx$$



$z^2+x^2+y^2=r^2$

$z=\sqrt{r^2-x^2-y^2}$

$$\iint_R f\,dA = \int_0^r \int_0^{\sqrt{r^2-x^2}} \underbrace{\sqrt{r^2-x^2-y^2}}_{x \text{ คงที่}}\, dy\, dx$$

ให้ $y = \sqrt{r^2-x^2}\sin(\theta)$

$$\sqrt{r^2-x^2-y^2} = \sqrt{r^2-x^2-(r^2-x^2)\sin^2\theta}$$
$$= \sqrt{(r^2-x^2)(1-\sin^2\theta)} = \sqrt{(r^2-x^2)\cos^2\theta}$$

ถ้า $y=0$ แล้ว $\theta = 0$

ถ้า $y = \sqrt{r^2-x^2}$ แล้ว $\theta = \frac{\pi}{2}$

$$dy = \sqrt{r^2-x^2}\cos(\theta)\,d\theta$$

$$\int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\,dy = (r^2-x^2)\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos(2\theta)+1}{2}$

$$= (r^2-x^2)\left(\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta)}{4}\,d(2\theta) + \int_0^{\frac{\pi}{2}} \frac{1}{2}\,d\theta\right)$$

$$= (r^2-x^2)\left(\frac{\sin(2\theta)}{4}\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + \frac{1}{2}\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\right) = \frac{\pi}{4}(r^2-x^2)$$

$$\iint_R f\,dA = \int_0^r\int_0^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\,dy\,dx = \frac{\pi}{4}\int_0^r r^2-x^2\,dx = \frac{\pi r^2}{4}x\Big|_{x=0}^{x=r} - \frac{\pi}{4}\frac{x^3}{3}\Big|_{x=0}^{x=r}$$

$$\left(\frac{1}{4}-\frac{1}{12} = \frac{3-1}{12} = \frac{2}{12}\right) \qquad = \frac{\pi}{4}r^3 - \frac{\pi}{12}r^3 = \frac{1}{6}\pi r^3$$

ปริมาตรของรูปทรงกลม $= 8\,\frac{1}{6}\pi r^3 = \frac{4}{3}\pi r^3$